NASA CONTRACTOR REPORT

NASA CR-1553

CONTROL OF LINEAR SYSTEMS WITH LARGE TIME DELAYS IN THE CONTROL

by Ronald E. Foerster

Prepared by
STANFORD UNIVERSITY
Stanford, Calif.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1970

NASA CR-1553

CONTROL OF LINEAR SYSTEMS WITH LARGE TIME DELAYS IN THE CONTROL

By Ronald E. Foerster

Issued by Originator as SUDAAR Report No. 376

Prepared under Grant No. NGL-05-020-007 by
STANFORD UNIVERSITY
Stanford, Calif.

for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

ABSTRACT

This study was motivated by the attitude-control problem from earth of a satellite in deep space. The main feature of this control problem is the presence of a large time delay in the control function due to the long times required to transmit the control signals. Time delays in the control function also arise in many other types of control problems, and thus the attitude-control problem is merely used to illustrate the influence which time delay has upon system behavior. It is assumed that the time delay is constant in magnitude and identical in magnitude for each component of the control vector. Linear systems are considered throughout this paper, but it is pointed out that much of what is said applies to non-linear systems as well.

The most common effect of a time delay in the control is the degradation of the system behavior with respect to its delay-free behavior. This is demonstrated by considering a second-order problem with bounded control. This degradation in system performance is due primarily to the inherent uncontrollability of systems with control delays in the time interval $[t_o, t_o + t_d]$ where t_o is the initial time and t_d is the magnitude of the time delay. A theorem to this effect is presented.

Optimal control of systems with time delay in the control is discussed by first considering the analogous delay-free system and then utilizing the results of the controllability discussion to pose an analogous optimization problem for time-delay systems. It is shown that the necessary conditions for optimality of time-delay systems are identical to those of the analogous delay-free system after time $t_o + t_d$. The significance of this fact is that the optimal control law for the time-delay system is a function of the predicted state, $x(t + t_d)$, instead of the present state, $x(t)$. Several examples are considered in order to demonstrate these ideas.

ACKNOWLEDGMENT

The author is indebted to his advisor, Professor Irmgard Flügge-Lotz, for her encouragement and guidance during the course of this research. The author also thanks Professor Arthur E. Bryson, Jr. for his invaluable assistance during the latter stages of this research.

This research was supported by the National Aeronautics and Space Administration under Research Grant **NGL-05-020-007**. This support was greatly appreciated.

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

LIST OF ILLUSTRATIONS

SYMBOLS

A	(nxn) system matrix
B	(nxm) distribution-of-control matrix
C_i	i^{th} command point in the phase plane
C,δ	constants of integration
$d(\ )$	differential of ( )
e	(nx1) error vector
$F[x(t_f),t_f]$	(scalar) cost of being in state $x(t_f)$ at time t_f
f	switching function for 2^{nd}-order example
H	variational Hamiltonion
I	(nxn) integrand of controllability matrix
I_m	(mxm) identity matrix
J,J'	(scalar) performance index
$\bar{J}$	(scalar) augmented performance index
k	(mx1) optimal control vector in feedback form
k_1	coefficient of error rate in linear switching function
L	(scalar) cost function along the trajectory
m	dimension of control vector
N	(scalar) bound on control component magnitude
n	dimension of state vector
Q_1	(nxn) weighting matrix of final state
Q_2	(mxm) weighting matrix of control vector
q	number of terminal constraints
r	(nx1) reference input vector
r,α	polar coordinates of a point in oblique phase plane

γ	angle specification of oblique coordinate system
S_i	i^{th} switch point in the phase plane
s	Laplace variable
sgn	signum function; $\text{sgn } f = \frac{f}{\lvert f \rvert}$
t	time
t_1	time of last switching before reaching the origin
t_c	control time
t_d	magnitude of time delay
t_f	final time
U	set of admissible control-vector functions for the delay-free problem
u	(mxl) control vector for delay-free system
$u_o(\cdot)$	(mxl) initial state of the delay (see pg. 11)
V	set of admissible control-vector functions for the time-delay problem
v	(mxl) control vector for time-delay system
W	(nxn) controllability matrix
x	(nxl) state vector
x_f	(nxl) final state vector
$\delta(\ )$	variation of ( )
ζ	damping coefficient
θ	time between second command and switching for first command
λ	(nxl) vector of adjoint variables
ν	(qxl) vector of Lagrange multipliers
$\bar{\nu}$	$\sqrt{1 - \zeta^2}$
σ,τ	dummy variables of integration
τ_p	half-period of a periodic motion

Φ	(scalar) augmented cost function of final state
φ	(nxn) state transition matrix
ψ	(qxl) vector of terminal constraints
$(\dot{\ })$	differentiation with respect to time
$(\ )^T$	transpose of a matrix
$(\ )^{-1}$	inverse of a matrix
$(\ )_o$	initial value
$(\ )_{op}$	optimal value
$(\ )_i$	i^{th} component of a vector (see Appendix B)
$(\ )_L$	quantity on zero-command curve
$(\ )_p$	quantity associated with periodic motion
$\lvert\cdot\rvert$	absolute value function
$(\ )_x$	(lxn) vector of partial derivatives of a scalar with respect to the components of the (nxl) vector x

CHAPTER I
INTRODUCTION

1.1 PROBLEM FORMULATION

Small time delays in the control are present in almost every control problem due to the finite time required to transmit the control signal from one part of the system to another. Various approximate methods are available to handle these small delays when it appears that they have a significant effect upon system behavior. The effect of the delay is usually significant when its magnitude is appreciable compared to the natural period of oscillation of the plant being controlled. Then the approximate methods are no longer applicable and a different method of analysis must be used to study these systems. This paper presents the results of a study conducted on the control of linear systems which possess large time delays in the control function.

The class of systems with time delay considered in this paper is restricted by the following two assumptions:

Assumption 1.

The time delay is fixed in magnitude during operation of the control function.

Assumption 2.

Each component of a multi-dimensional control function possesses a time delay identical in magnitude to every other component.

An example of a linear system with time delay, which arises naturally in the aerospace field, is that of the linearized rèmote control from earth of a deep-space satellite. In particular, it may be desirable to control the attitude of an unmanned vehicle, upon which some detection device, such as a telescope, is mounted. Controlling the vehicle's attitude from earth allows greater freedom for exploration and reduces onboard computer requirements. One can easily formulate this control problem such that Assumptions (1) - (2) are satisfied. This study is highly motivated by the above attitude control problem, but the theory presented here is applicable to any problem involving a time delay in the control as long as the above assumptions are satisfied.

A more precise formulation of the control problem is now presented. Assume that the system dynamics are specified by the following set of n linear, first-order, ordinary differential-difference equations:

$$\dot{x}(t) = A(t)\, x\,(t) + B(t)\, u(t-t_d) \tag{1.1}$$

where

$x(t)$ is the nxl state vector,

$A(t)$ is a nxn matrix,

$B(t)$ is the nxm distribution-of-control matrix,

$u(t-t_d)$ is the mxl control vector, each component of which is delayed by the constant time delay t_d.

A block diagram of this system, as it pertains to the attitude control problem discussed above, is shown in Figure 1.1. It is noted here that the satellite attitude equations of motion can be written in the form of Equation (1.1) only after linearizing the original non-linear equations of motion and neglecting all inhomogeneous terms arising from gravity gradient torques. This latter step is justified by the fact that the satellite is in deep space where gravity gradient torques are assumed negligible.

In Figure 1.1, $r(\cdot)$ and $e(\cdot)$ are nxl vector functions of time. The reference signal, $r(\cdot)$, is the desired value of the state. Since a time of magnitude $t_{d/2}$ is required to transmit the value of the state from the satellite to earth, $r(\cdot)$ must be delayed for a time $t_{d/2}$ in order to form the error signal $e\,(t-{t_d}/2)$. The controller formulates a control signal on earth, knowing the error in the state at a time $t_{d/2}$ earlier, and then transmits this signal back to the satellite, resulting in another delay of magnitude $t_{d/2}$.

The problem is to construct a controller, subject to certain restrictions, such that $e(\cdot)$ goes to zero in some desirable fashion. When $r(\cdot)$ is identically zero, this control problem reduces to a regulator control problem. If $r(\cdot)$ is constant or slowly varying, the solution

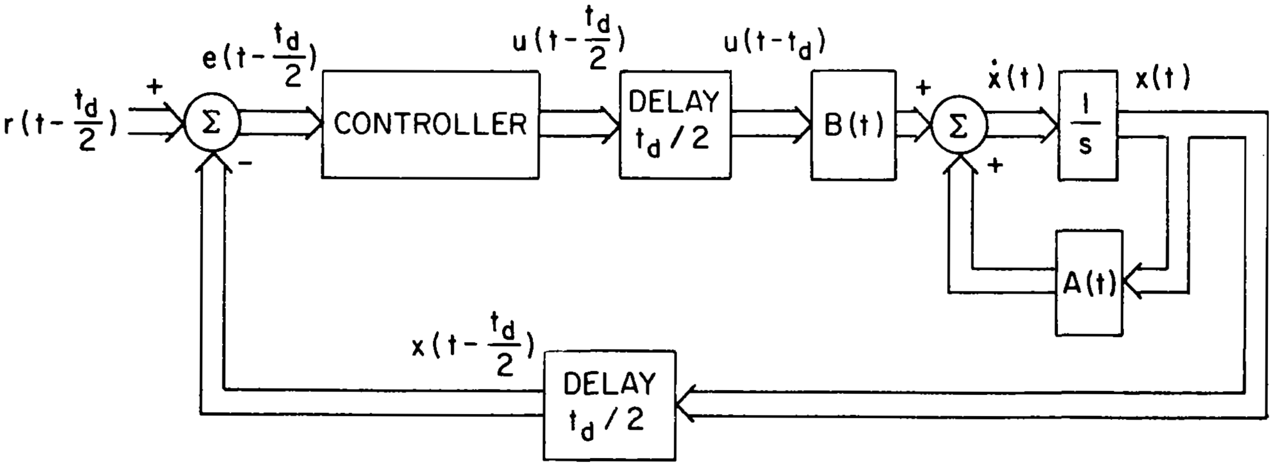


Figure 1.1. Block Diagram for a Deep-Space Attitude-Control Problem Assuming Linear Vehicle Dynamics.

of the regulator problem is an obvious candidate for the control function and is thus the objective in this study.

The total delay in Figure 1.1, between the measurement of the state, $x(t)$, and the action of the resulting control signal, is of magnitude t_d. One final simplification of the system shown in Figure 1.1 will be made using this fact. The two delays in Figure 1.1 are combined into a single delay of magnitude t_d. The resulting block diagram is shown in Figure 1.2.

The simplification of the original system made in Figure 1.2 removes the direct analogy between the block diagram and the deep-space attitude control problem. The controller design, however, can be constructed from either of these systems, even when $r(\cdot)$ is not identically zero. In particular, the controller in Figure 1.2 can be used in Figure 1.1, after a time shift of magnitude $t_{d/2}$, such that the two systems have identical behavior. Designing the controller for the system shown in Figure 1.2 is thus sufficient for solving the deep-space attitude-control problem.

The assumption that the system of interest is linear in the state and control is most convenient in the discussion on controllability. It is not essential in the discussion on system optimization. Possible generalizations of the system shown in Figure 1.2 are considered in Chapter VI.

1.2 PREVIOUS RESULTS

The attitude control problem discussed in Section 1.1 is the primary motivation for this study. Sabroff [1]* presents a sound case for earth-based control of satellites in deep space, assuming the control problem created by the time delay can be solved. Adams [2] and Martin [3] considered the problem of remote control, from earth, of a lunar unmanned

*Numbers in brackets, [·], refer to references given at the end of this paper.

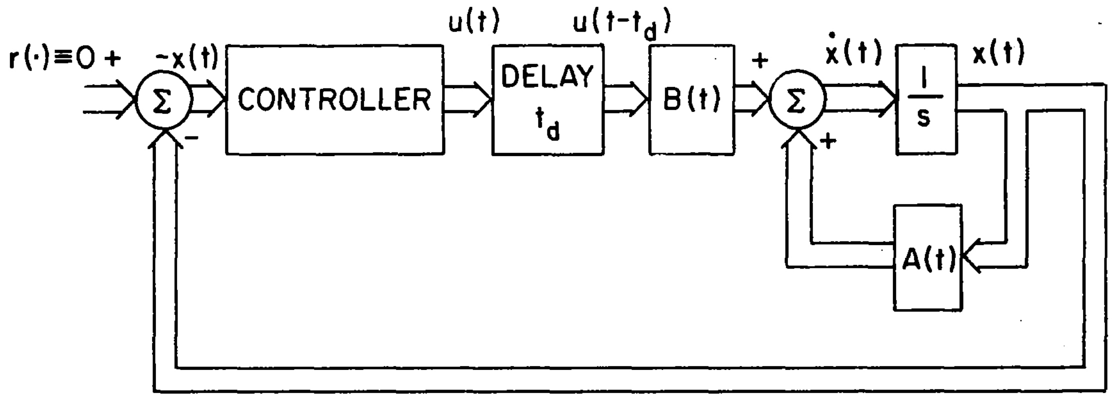


Figure 1.2. Simplified Block Diagram of System Shown in Figure 1.1.

surface vehicle. Both of these studies were experimental and revealed some of the problems created by time delays, but neither satisfactorily solves this control problem. Flügge-Lotz [4] considered the effect of time delay resulting from relay imperfections in bang-bang control problems. The results obtained, however, are not valid when the time delay is large, even for the simple control problems considered there.

Fuller [5] summarizes the recent theoretical developments in the optimal control of systems with time delay in the control. The list of references given by Fuller is fairly complete and will not be repeated here. Very briefly, it can be said that most researchers of optimal control of systems with a control delay have concluded that the optimal control law must be based upon the predicted future of the state. Some researchers recognized the fact that if an optimal feedback control law is desired, it must have a functional dependence upon the control history over a finite span of time. This segment of the control history is termed the "state of the delay" by Fuller. As Fuller points out, most of the "optimal" feedback control laws derived in the literature are, in fact, sub-optimal since the "state of the delay" was not properly accounted for by the authors.

Fuller recognized the basic relationships between optimal control laws for time-dealy and analogous delay-free systems. Without considering the necessary conditions for optimality, he conjectured that the optimal feedback control law for the time-delay system is the same as the optimal feedback control law for the delay-free system when based upon the predicted state, $x(t + t_d)$. He concerns himself with bang-bang control problems and the conjecture is true if t_d is restricted in magnitude. Fuller did not recognize that if t_d is sufficiently large, an optimal feedback discontinuous control law cannot be obtained for the time-delay system, even if it is known for the delay-free system. Most of Fuller's analysis presupposes that the control law for the delay-free system is known explicitly. This is highly restrictive and, using his analysis when this information is available, reduces the insight into the control problem one obtains by studying the control of the delay-free system. It should also be noted that Fuller's optimal feedback control law is not optimal at all when an unknown disturbance acts on the system. This fact is true for any optimal control law designed for a system with time delay in the control.

Ichikawa [6] approached this control problem from a different point of view. He considered a fairly general optimization problem for a non-linear system with delays in both the control and the state. The performance criterion was completely general and end-point equality constraints were considered. By writing the finite number of difference-differential equations as an infinite number of ordinary differential equations and then applying Pontryagin's maximum principle, he obtains a set of necessary conditions for optimality. These conditions lead to controllers which are functionally dependent upon the state of the delay. Considering only the delay in the control, the necessary conditions for optimality derived by Ichikawa can be easily derived, for a large class of problems, without considering any infinite dimensional system of ordinary differential equations. Also, applying his necessary conditions for optimality to obtain an optimal control law does not allow one to make full use of his knowledge about the optimal control of the analogous delay-free system. Nevertheless, it is interesting to compare the necessary conditions obtained here, for a more restricted class of problems, with those of Ichikawa.

1.3 ORGANIZATION OF WORK

Most of the analysis presented in this paper pertains to optimal control of linear systems with time delay in the control. In Chapter II, however, the general effect of time delay in the control upon system behavior is discussed. The "state of the delay" is defined and an analogy to the "state" is given which adds insight into the control problem. A second-order system with bounded control is then considered in order to demonstrate the adverse effects which a time delay in the control can have upon system behavior. Both linear and minimum-time switching functions are considered. The minimum-time problem for the second-order system is reconsidered in Chapter V. It is emphasized here that these simple examples are presented to illustrate basic features of control delay and are not intended to solve more complicated problems such as the 6th-order satellite attitude-control problem.

The remainder of the report deals with general optimal control of time-delay systems. In Chapters III and IV a set of necessary conditions for optimality are obtained. Implementation of these necessary conditions is discussed in Chapter V.

The sole purpose of Chapter III is to present the basis upon which a time-delay analogue of the delay-free system and optimization problem is constructed. Controllability requirements are discussed and a fairly general optimization problem is posed. The necessary conditions for optimality of the delay-free system are then presented without proof. Except for several modifications of the definition of controllability, which were made so that the definition could also be applied to time-delay systems, the results of this chapter are well known.

The structure of Chapter IV parallels that of Chapter III except now time-delay systems are considered. Controllability plays an important role in the discussion of optimal control of time-delay systems, and is therefore considered in some detail. The necessary conditions for optimality, for the optimization problem posed in this chapter, are obtained from the necessary conditions for optimality of the analogous delay-free system given in Chapter III. These conditions are also derived in Appendix A by means of the calculus of variations. It was felt, however, that relating the delay-free and the time-delay problems would give more insight into how to utilize the solution of one problem to obtain the solution to the other problem.

The calculation and implementation of optimal control laws from the necessary conditions for optimality given in Chapter IV is the subject of Chapter V. The general relationships between the control laws for the delay-free and time-delay systems are first discussed. Two examples are then considered. The first problem demonstrates how the necessary conditions for optimality can be used directly to obtain an optimal control law for an n^{th}-order system with unbounded control and a quadratic performance index. The second problem shows how the delay-free solution can be utilized to solve the time-delay analogue of the second-order, minimum-settling-time example discussed in Chapter II.

Finally, the basic conclusions drawn in this work are summarized in Chapter VI, and some possible generalizations are suggested.

CHAPTER II

EFFECT OF TIME DELAY UPON SYSTEM BEHAVIOR

2.1 THE STATE OF THE DELAY

The presence of a large time delay in the control signal of a linear system has a significant effect upon the dynamics of the system. This effect is largely attributed to the existence of a quantity which A. T. Fuller [5] has termed "the state of the delay". This terminology is quite appropriate. Just as knowledge of the state, $x(t)$, is necessary in order to completely specify a delay-free system, the "state of the delay" must be known, in addition to $x(t)$, in order to completely specify a system with a delay in the control.

It is assumed that the system dynamics are given by Equation (1.1) and that the initial state, $x(t_o) = x_o$, is specified. The solution of Equation (1.1) may then be deduced from the solution of the analogous delay-free equation (Equation (1.1) with $t_d = 0$):

$$x(t) = \varphi(t,t_o)x_o + \int_{t_o}^{t} \varphi(t,\tau)B(\tau)u(\tau-t_d)d\tau, \qquad t \geq t_o \tag{2.1}$$

were $\varphi(\cdot,\cdot)$ is the state transition matrix for Equation (1.1). Now, $u(\cdot)$ is the control signal generated by the controller in Figure (1.2) and hence is initiated at time t_o. Since this signal is delayed a time t_d, the control initiated at time t_o won't be "felt" until time $t_o + t_d$. This is easily seen by evaluating Equation (2.1) at $t = t_o + t_d$ and changing variables in the integral:

$$x(t_o + t_d) = \varphi(t_o + t_d, t_o)x_o + \int_{t_o-t_d}^{t_o} \varphi(t_o + t_d, \sigma + t_d)B(\sigma + t_d)u(\sigma)d\sigma \tag{2.2}$$

Thus, the function $u(\sigma), \sigma\epsilon[t_o - t_d, t_o]$, is needed in order to calculate $x(t_o + t_d)$, and hence the state, $x(t)$, for any $t > t_o$. Note that this function is not part of the control function generated by the controller after time $t = t_o$.

The state of the delay may now be defined in terms of the above notation.

Definition 2.1

The state of the delay at time t of the system described by Equation (1.1) is the vector function $u(\tau), \tau\epsilon[t - t_d, t]$. If the initial time is t_o, the initial state of the delay, $u_o(\tau)$, $\tau\epsilon[t_o, t_o + t_d]$, is given by

$$u_o(\tau) = u(\tau - t_d),\ \tau\epsilon[t_o, t_o + t_d] \tag{2.3}$$

Thus, from Definition 2.1, the state of the delay is merely that portion of the control history generated during the last t_d units of time. Fuller [5] likens systems with a control delay to a magnetic tape moving at constant speed between two reels upon which data, $u(t)$, is recorded at one reel and from which data, $u(t - t_d)$, is erased at the other reel. The state of the delay at time t is then represented by all of the data on the tape at time t. This analogy may be helpful in visualizing the effect which a delay in the control has upon system behavior. The fact that one cannot alter the data on the tape, once recorded, suggests that systems with a control delay are uncontrollable during the time interval $[t, t + t_d]$.

Before considering controllability and optimal control, a second-order system is investigated in order to demonstrate the effect which a control delay, and its state, has upon system performance.

2.2 SECOND-ORDER EXAMPLE WITH BOUNDED CONTROL

A second-order system with control delay is considered in this section. Let $e(t)$ be a scalar error signal which satisfies the following second-order differential-difference equation:

$$\ddot{e}(t) + 2\zeta\dot{e}(t) + e(t) = u(t - t_d) \tag{2.4}$$

In Equation (2.4), time t has been non-dimensionalized with the natural frequency of the system. Thus both t and t_d are measured in radians. Also, $u(\cdot)$ is the scalar control function and ζ is the positive damping coefficient of the system. Since $e(t)$ is designated as an error signal, the control problem in this example is a regulator problem.

Assume that the state, $(e(t),\dot{e}(t))$, is zero prior to the action of an impulsive disturbance upon the system at time t_o. This disturbance gives the system a non-zero initial state, $(e_o,\dot{e}_o)$, and since $u(t - t_d) = 0$, $t\epsilon[t_o,t_o + t_d]$, the initial state of the delay is identically zero.

Now assume that the magnitude of the control is bounded:

$$|u(t - t_d)| \le N, \qquad t \ge t_o \tag{2.5}$$

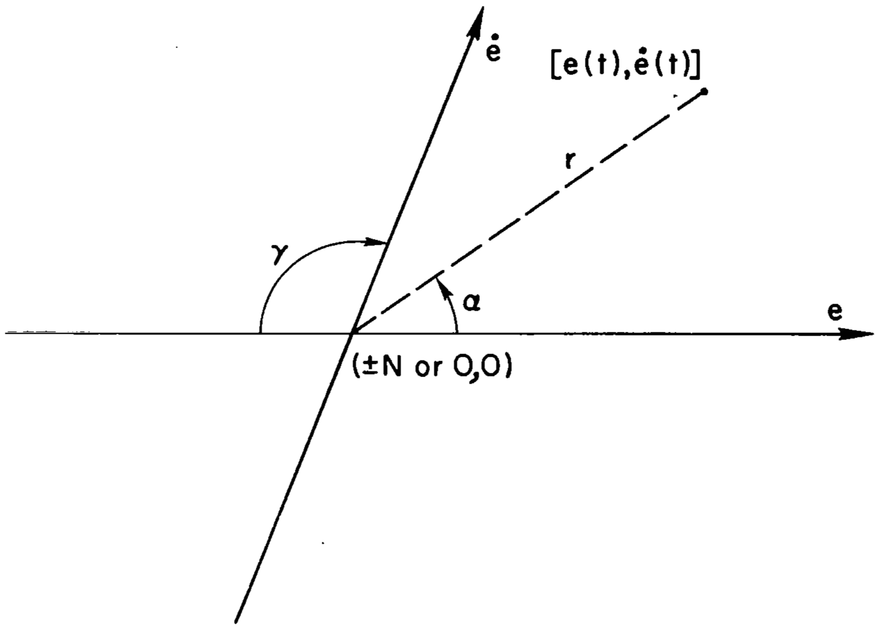
where N is a positive constant. This control constraint suggests the possibility of choosing $u(\cdot)$ to have a bang-bang structure. This type of controller is not only easily implemented, but also proves to be optimal in a large class of optimization problems. We shall assume, therefore, that the control can be specified by

$$\begin{aligned} u(t - t_d) &= -N\mathrm{sgn}[f(e(t - t_d),\dot{e}(t - t_d))], \quad & t \ge t_o + t_d \\ &= 0 \ , & t_o \le t < t_o + t_d \end{aligned} \tag{2.6}$$

where $\mathrm{sgn}\ x = \frac{x}{|x|}$ and $f(\cdot,\cdot)$ is the switching function in the state variables. Writing $u(\cdot)$ in this way limits this discussion to feedback control laws.

The trajectories of Equation (2.4), using the control law in Equation (2.6), are logarithmic spirals about $(\pm N \text{ or } 0,0)$ in the $(e,\dot{e})$-phase plane when an oblique coordinate system is used [7]. This coordinate system, along with the necessary defining relationships for the trajectories, is shown in Figure 2.1.

Thus, from the defining equations in Figure 2.1, it is seen that r is a converging logarithmic spiral when ζ is positive and α varies linearly with the dimensionless time. Note that since $\Delta\alpha = -\sqrt{1 - \zeta^2}\,\Delta t$, the dimensionless time delay t_d results in an angular displacement in the phase plane of magnitude $\sqrt{1 - \zeta^2}\, t_d$ radians. These facts are used below in the construction of trajectories for two particular switching functions.



DEFINING EQUATIONS

$[e(t),\dot{e}(t)]$	—	$\ddot{e}(t) + 2\zeta\dot{e}(t) + e(t) = \pm N$ or 0
γ	—	$\cos\gamma = -\zeta$, $\sin\gamma = \sqrt{1-\zeta^2}$
r	—	$r = Ce^{-\zeta t}\sin\gamma$
α	—	$\alpha = \delta - \sqrt{1-\zeta^2}\,t$
C,δ	—	Determined by $e(t_o),\dot{e}(t_o)$

Figure 2.1. Oblique Coordinate System and Defining Equations for the Trajectories of the Second-Order Example.

2.2.1 LINEAR SWITCHING FUNCTION

The most simple switching function to implement in the construction of bang-bang controllers is the linear switching function:

$$f[e(t),\dot{e}(t)] = e(t) + k_1\dot{e}(t), \qquad t \geq t_o \tag{2.7}$$

where k_1 is a scalar constant. When $k_1 > 0$, this switching function causes the trajectories of the delay-free system to converge to the origin, ending in a chatter motion when the relay is non-ideal [4].

The introduction of a time delay, t_d, into the control law causes the switching function to become

$$f[e(t - t_d),\dot{e}(t - t_d)] = e(t - t_d) + k_1\dot{e}(t - t_d), \qquad t \geq t_o + t_d \tag{2.8}$$

when the command curve is given by Equation (2.7). The effect of the time delay is a rotation of the switching curve in Equation (2.7) and a division of the single switching curve into two parallel switching curves. This is clear when the switching function in Equation (2.8) is written in terms of $(e(t),\dot{e}(t))$ [4, Eq. 138 in modified form]:

$$\begin{aligned} f[e(t - t_d),\dot{e}(t - t_d)] &= e(t)\left[\cos\bar{\nu}t_d + \frac{k_1\sin\bar{\nu}t_d}{\bar{\nu}} - \frac{\zeta\sin\bar{\nu}t_d}{\bar{\nu}}\right] \\ &+ \dot{e}(t)\left[k_1\cos\bar{\nu}t_d - \frac{\sin\bar{\nu}t_d}{\bar{\nu}} + \frac{k_1\zeta\sin\bar{\nu}t_d}{\bar{\nu}}\right] \mp \operatorname{sgn}(f[e(t),\dot{e}(t)]) \times \\ &\left[\cos\bar{\nu}t_d + \frac{k_1\sin\bar{\nu}t_d}{\bar{\nu}} - \frac{\zeta\sin\bar{\nu}t_d}{\bar{\nu}} - \exp^{-\zeta t_d}\right] \end{aligned} \tag{2.9}$$

where

$$\bar{\nu} = \sqrt{1 - \zeta^2} \tag{2.10}$$

Also, it is shown in [4] that if a periodic motion exists, then the half-period, τ_p, of this motion is the solution of the following transcendental equation:

$$\begin{aligned} &[\sin(\bar{\nu}(t_d - \tau_p) + \gamma) + k_1\sin\bar{\nu}(t_d - \tau_p)] + \exp^{-\zeta\tau_p}[\sin(\bar{\nu}t_d + \gamma) \\ &+ k_1\sin\bar{\nu}t_d] = \bar{\nu}\exp^{-\zeta t_d}[\cos\bar{\nu}\tau_p + \cosh\zeta\tau_p] \end{aligned} \tag{2.11}$$

and the switch points of this periodic motion, $(e_p, \dot{e}_p)$, are given by

$$e_p = \mp \operatorname{sgn}[-f(e(t),\dot{e}(t))] \frac{\frac{\zeta}{\bar{\nu}} \sin\bar{\nu}\tau_p - \sinh\zeta\tau_p}{\cos\bar{\nu}\tau_p + \cosh\zeta\tau_p} \tag{2.12}$$

$$\dot{e}_p = \pm \operatorname{sgn}[-f(e(t),\dot{e}(t))] \frac{\frac{\sin\bar{\nu}\tau_p}{\bar{\nu}}}{\cos\bar{\nu}\tau_p + \cosh\zeta\tau_p} \tag{2.13}$$

It is important to note here that the above results are valid only when a double-command does not occur during the motion of the system.*

A double-command occurs when a second command is given before switching for the first command can be executed. This will always happen when t_d is very large $(t_d >> \pi \text{ rad})$. Martin [3] considered control of this system by means of reverse switching ($+N\operatorname{sgn} f(\cdot,\cdot)$ in Equation (2.6) instead of $-N\operatorname{sgn} f(\cdot,\cdot)$). When this type of switching is used, double commands can occur for any size t_d if the initial conditions for the trajectory lie in a certain region of the phase plane. When double commands occur, Equation (2.12) and Equation (2.13) still give $(e_p, \dot{e}_p)$ for a periodic motion but Equation (2.11) for τ_p is replaced by

$$[\sin(\bar{\nu}(t_d - \tau_p) + \gamma) + k_1 \sin\bar{\nu}(t_d - \tau_p)] + \exp^{-\zeta\tau_p}[\sin(\bar{\nu}t_d + \gamma) + k_1 \sin\bar{\nu}t_d]$$

$$= 2\exp^{-\zeta\tau_p}[\cos\bar{\nu}\tau_p + \cosh\zeta\tau_p][\sin(\bar{\nu}(t_d - \tau_p) + \gamma)$$

$$+ k_1 \sin\bar{\nu}(t_d - \tau_p) - \frac{\bar{\nu}}{2} \exp^{-\zeta(t_d - \tau_p)}] \tag{2.14}$$

Also, the equation for the switching curve is altered to become

$$f[e(t - t_d), \dot{e}(t - t_d)] = e(t)[\sin(\bar{\nu}t_d + \gamma) + k_1 \sin\bar{\nu}t_d] \tag{2.15}$$

$$-\dot{e}(t)[k_1 \sin(\bar{\nu}t_d - \gamma) + \sin\bar{\nu}t_d] \mp \operatorname{sgn}(f[e(t),\dot{e}(t)]) \Big\{ [\sin(\bar{\nu}t_d + \gamma)$$

(continued)

*This fact was overlooked in [4].

$$
\left. \begin{aligned} &+ k_1 \sin \bar{\nu} t_d + \bar{\nu} \exp^{-\zeta t_d}] - 2\exp^{-\zeta(t_d - \theta)}[\sin(\bar{\nu}\theta + \gamma) \\ &+ k_1 \sin \bar{\nu}\theta] \end{aligned} \right\} = 0 \tag{2.15}
$$

where θ is the normalized time elapsed after the second command is given until switching for the first command is executed. This quantity is a function of the initial state of the system and thus Equation (2.15) does not give a feedback representation of the switching points for this problem. The derivations of Equation (2.14) and Equation (2.15) are quite tedious, but follow the straightforward approach used to derive Equation (2.9) and Equation (2.11) in [4].

Example trajectories having double commands as the result of a large time delay and as the result of reversed switching are shown in Figure (2.2) and Figure (2.3), respectively. The command points are labeled C_i and the corresponding switch points are labeled S_i. The trajectories for the analogous delay-free system, possessing the same initial state, are indicated by the dashed lines in these figures. It is observed that the presence of double commands invalidate the expression for the switching curve in Equation (2.9). Equation (2.15) is verified only after a lengthy calculation. Also note, particularly in Figure (2.3), that double commands can result in trajectories with highly undesirable properties. Furthermore, the sequence of switch points along a trajectory with double commands can be determined only in an open-loop sense since the sequence is highly sensitive to the initial state. This initial state sensitivity appears in the quantity θ in Equation (2.15). Trajectory design for a given initial state is, therefore, extremely difficult. Double commands should be avoided whenever possible.*

Now assuming the non-existence of double commands, the validity of the expression for the switching curve in Equation (2.9), and the conditions for periodic motion in Equations (2.11)-(2.13), is demonstrated in Figure (2.4) and Figure (2.5). Figure (2.4) shows an example trajectory which has desirable properties when t_d is zero but which ends in a large amplitude limit

*The remarks made for double commands are equally valid for multiple-commands of any order.

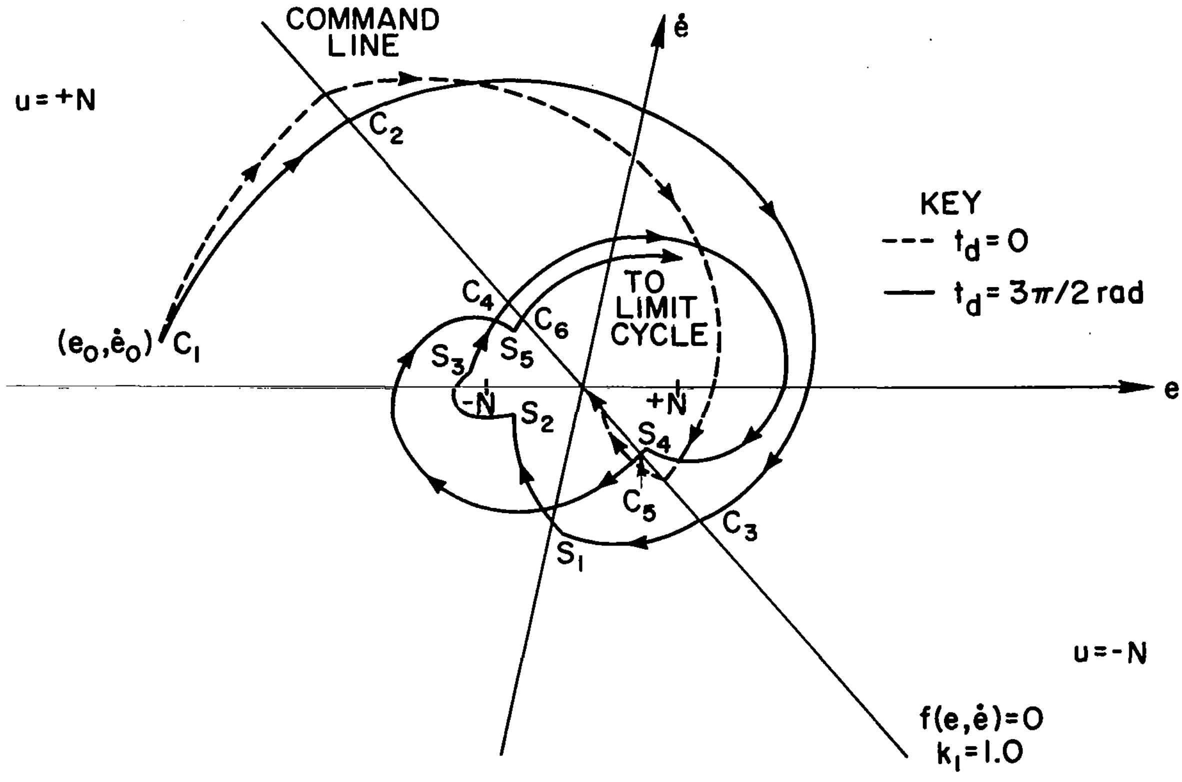


Figure 2.2. Example Trajectories for Second-Order System with Damping Coefficient $\zeta = 0.2$ Using a Linear Command Curve ($k_1 = 1.0$): Double Commands Due to Large Time Delay.

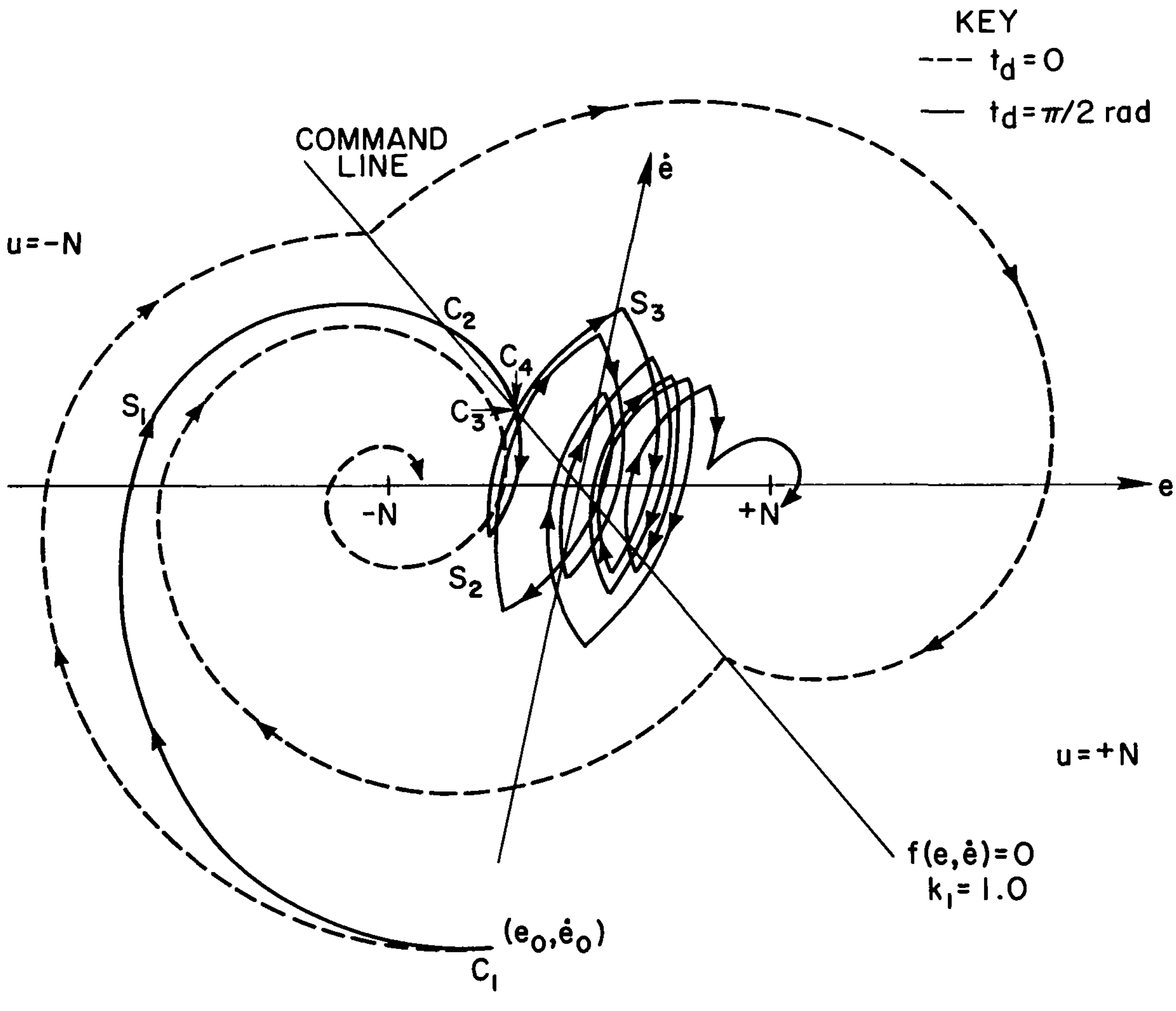


Figure 2.3. Example Trajectories for Second-Order System with $\zeta = 0.2$ Using a Linear Command Curve ($k_1 = 1.0$): Double Commands Due to Reversed Switching.

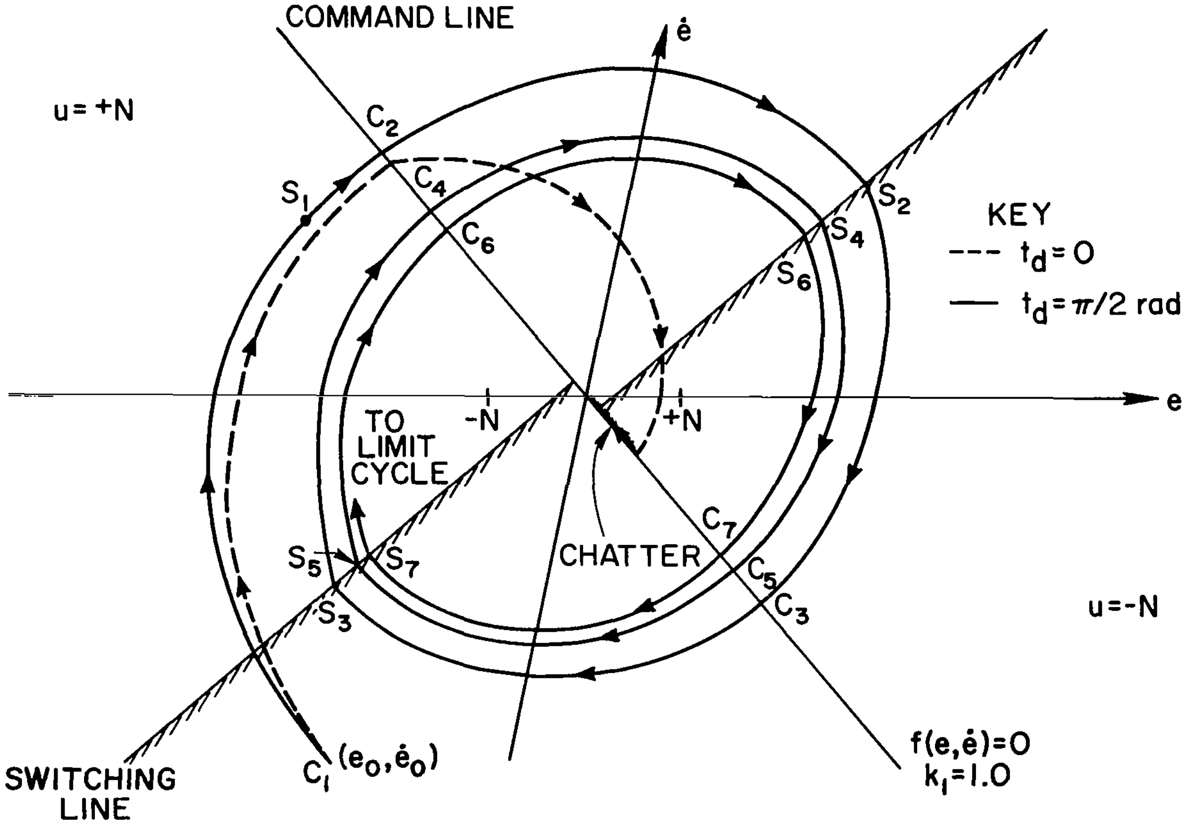


Figure 2.4. Example Trajectories for Second-Order System with $\zeta = 0.2$ Using a Linear Command Curve ($k_1 = 1.0$): Single Commands.

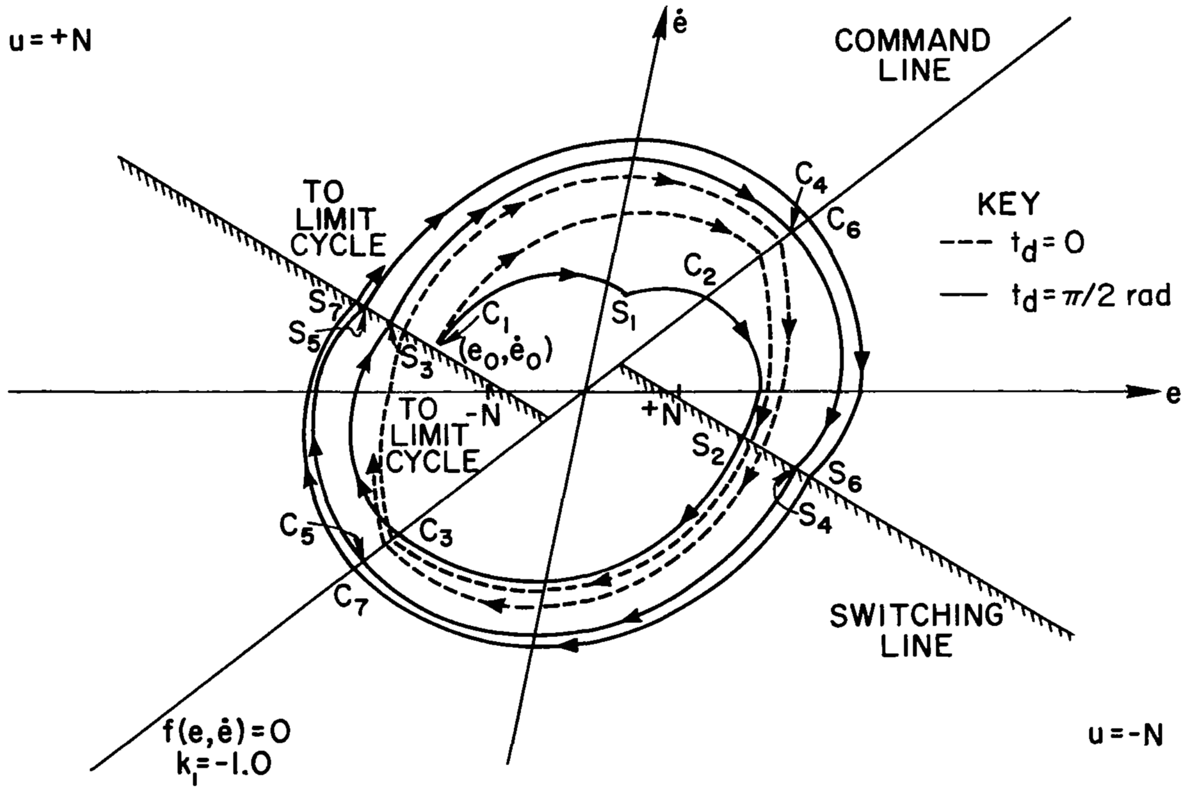


Figure 2.5. Example Trajectories for Second-Order System with $\zeta = 0.2$ Using a Linear Command Curve ($k_1 = -1.0$): Single Commands.

cycle about the origin when t_d is large. For the case shown in Figure (2.4), application of Equations (2.11) - (2.13) gives $\tau_p \approx 156^\circ$ and $(e_p,\dot{e}_p) = (\pm1.83N,\pm1.78N)$. Similarly, $\tau_p \approx 200^\circ$ and $(e_p,\dot{e}_p) = (\pm2.78N,\mp.96N)$ for the trajectory shown in Figure (2.5). This latter limit cycle is larger in amplitude than the limit cycle obtained with no time delay. This is a consequence of the discontinuity of the switching line at the origin caused by the delay in the control. Note that the trajectories shown in Figure (2.5) are similar to those obtained by using reverse switching, positive k_1, and no time delay. It should also be noted that as long as t_d is non-zero the steady-state motion of the system is a finite amplitude limit cycle.

The intent of the above examples was to show the effect which a time delay in the control signal has upon the trajectories of the second-order system when a linear command function is used. Similar discussions can be conducted for other command functions, resulting in many of the same conclusions drawn from the above examples. The presence of a finite amplitude limit cycle, and generally undesirable behavior in the neighborhood of the origin, are characteristics of trajectories that time delays in the control create. One other command function is considered in the next section since it will be relevant in the discussion on optimization, and since it further demonstrates the effects caused by time delays.

2.2.2 MINIMUM TIME SWITCHING FUNCTION FOR A DELAY-FREE SYSTEM USED FOR A SYSTEM WITH DELAY IN THE CONTROL

Bushaw [8] determined the switching function which gives minimum time trajectories from an initial state to the origin for the second-order example (Equation (2.4)) with t_d equal to zero. It is difficult to express this switching function, $f(e(t),\dot{e}(t))$, explicitly. In the $(e,\dot{e})$-phase plane, the switching function is constructed from a sequence of logarithmic spiral segments having centers lying on the e-axis. For the special case of zero damping, the spiral segments are semi-circles of constant radius. This case is illustrated in Figure 2.6 and Figure 2.7.

Example trajectories of the minimum time solution to the control problem for the delay-free system are shown by dashed lines in Figure 2.6 and Figure 2.7. Since $\zeta = 0.0$ in these figures, the trajectories are

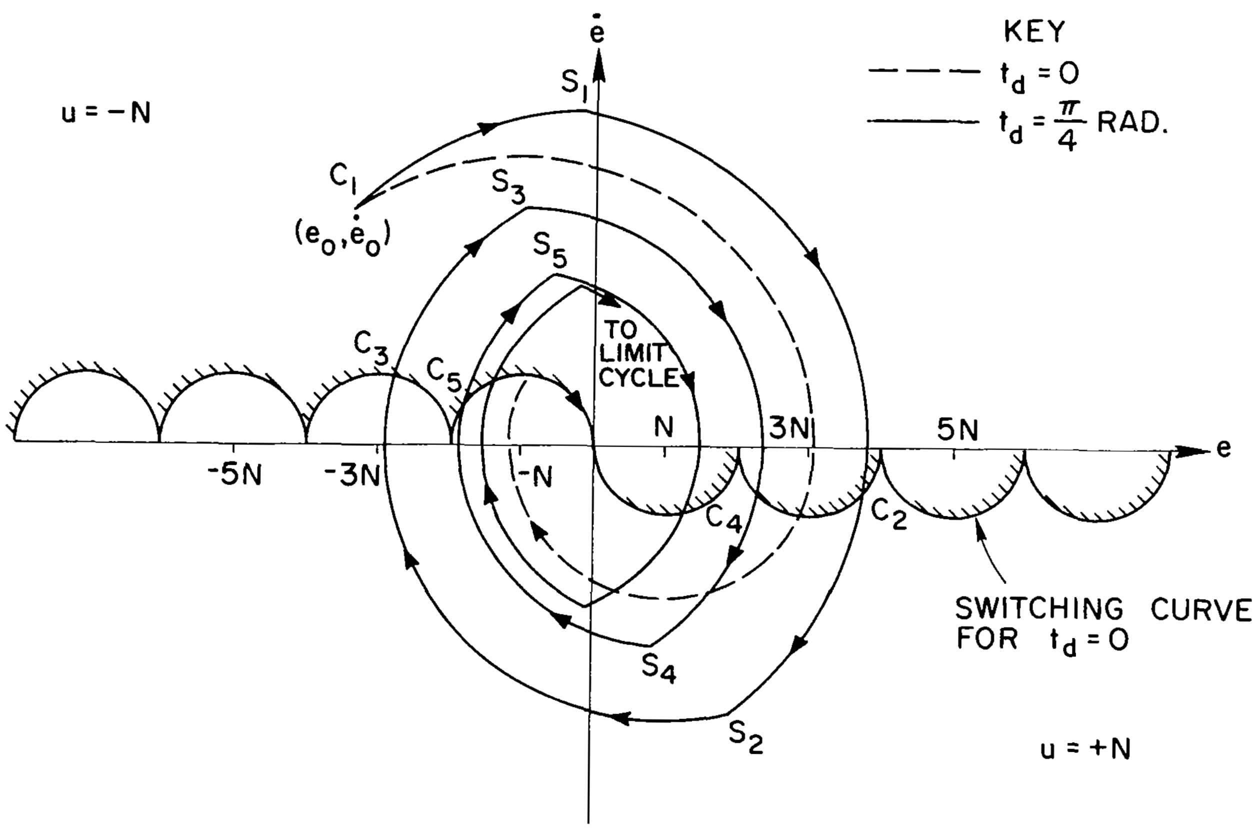


Figure 2.6. Sample Trajectories for a Second-Order Example with Zero Damping Using Bushaw's Minimum-Time Switching Curve: Single Commands.

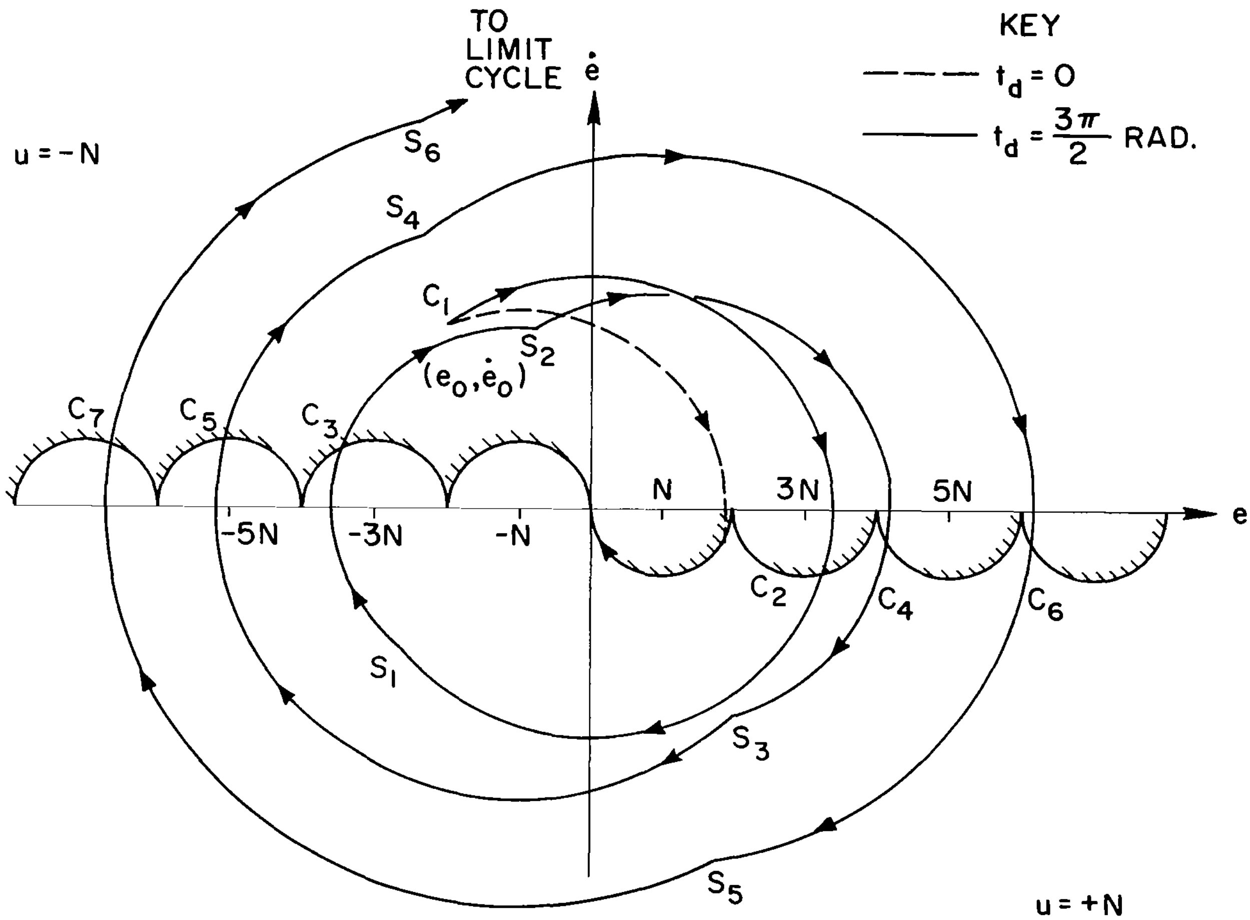


Figure 2.7. Sample Trajectories for a Second-Order Example with Zero Damping Using Bushaw's Minimum-Time Switching Curve: Double Commands.

composed of circular arcs, centered about $(\pm N,0)$*. Also shown in these figures are trajectories which result when this minimum time switching function is used as a command function for the second-order example with time delay. A double-command trajectory is shown in Figure 2.7, whereas only single commands occur on the trajectories shown in Figure 2.6. A finite-amplitude limit cycle is reached in Figure 2.6 and divergent motion, which possibly results in a very large-amplitude limit cycle, is the consequence of the time delay in Figure 2.7. In both of these examples, the introduction of time delay in the control has drastically altered the behavior of the system trajectories. The control law must, therefore, be altered in order to duplicate the delay-free system behavior when time delay is present.

This example will be considered again in Chapter V, following the discussion on optimal control in Chapter III and Chapter IV. At that point the switching logic will be presented which gives minimum time trajectories for the second-order example with time delay, assuming only single commands occur.

*See the Defining Equations in Figure 2.1.

CHAPTER III

OPTIMAL CONTROL OF DELAY-FREE LINEAR SYSTEMS

3.1 SYSTEM DESCRIPTION

The system of interest in this chapter is described by the following set of n linear, first-order, ordinary differential equations:

$$\dot{x}(t) = A(t)\, x(t) + B(t)u(t), \qquad x(t_o) = x_o \tag{3.1}$$

where

$x(t)$ is the $n \times 1$ state vector,

$u(t)$ is the $m \times 1$ control vector,

$B(t)$ is the $n \times m$ distribution-of-control matrix,

$A(t)$ is a $n \times n$ matrix.

This system is the delay-free analogue of the time-delay system specified by Equation (1.1). The assumption that the system of interest can be described by Equation (3.1) implies that the origin, which is the point of interest in a regulator problem, is an equilibrium point of the uncontrolled system. When the origin is not an equilibrium point (as is the case when a non-zero forcing function is included in Equation (3.1)), a non-zero control function is needed to hold the origin once it is attained. This problem will not be discussed in detail in this paper.

Since Equation (3.1) contains no time delay, standard techniques may be applied to determine a satisfactory control function, $u(t)$, with respect to a chosen performance index [4,7,9,11,12]. The purpose of this chapter is to present some known results for systems described by Equation (3.1). These results will then be used in Chapter IV to analyize linear systems possessing a time delay in the control function. The relationships between systems which possess a time delay, and those which do not, are then discussed in Chapter V.

The concept of controllability is important in the discussion of systems possessing time delays. A brief discussion of this concept, as it pertains to delay-free systems, is thus presented in the next section.

3.2 CONTROLLABILITY OF DELAY-FREE SYSTEMS

Before presenting an explicit definition of system controllability, the following definitions are made in order to clarify this concept when applying it to systems with time delay*:

Definition 3.1

A state $x(t_o) = x_o$ is transferred to a state $x(t_f) = x_f \neq x_o$ by a control $u(t)$, $t_o \leq t < \infty$, if $x(t) = x_f$ for all $t \geq t_f$.

Definition 3.2

The control time, t_c, is the time required to transfer a state from one value to another.

A general definition of controllability may now be given which is valid for systems possessing time delays, as well as those which do not.

Definition 3.3

A state $x(t_o) = x_o$ is controllable at time t_o in the control time $t_c (=t_f - t_o)$ if there exists some finite time $t_f > t_o$ and some control vector $u(t)$, $t \epsilon (t_o, \infty)$, which transfers $x_o \neq x_f$ to the state $x(t_f) = x_f$ at time t_f. (If every state x_o is controllable in any control time $t_c > 0$, then the system associated with x_o is said to be completely controllable.)

The final state, x_f, may be equated to zero by a transformation of coordinates without loss of generality. This is done, for convenience, throughout the remainder of this discussion. It should also be noted that the concepts of "transfer" and "control time" expressed in Definitions (3.1) - (3.2) need not enter the discussion on controllability of systems described by Equation (3.1). Assuming the availability of an unbounded control, the control time, t_c, for a controllable state, x_o, is completely arbitrary. Also, from Equation (3.1) and the fact that the origin is an equilibrium point of the uncontrolled system, the state $x(t)$ is zero

*Definitions 3.1 - 3.2 are similar to those given in [9], pg. 71.

for all $t > t_f$ if and only if $u(t) = 0$, $t \epsilon [t_f, \infty]$ and $x(t_f) = 0$ where t_f ia arbitrary since t_c is arbitrary. These remarks may not be made about systems possessing time delays, and thus "transfer" and "control time" are incorporated into the definition of controllability.

Now let $\varphi(\cdot,\cdot)$ be the state transition matrix for Equation (3.1)* and define the "controllability matrix" of this equation to be

$$W(t_o,t_f) \equiv \int_{t_o}^{t_f} \varphi(t_o,\tau)B(\tau)B^T(\tau)\varphi^T(t_o,\tau)d\tau. \tag{3.2}$$

The test for system controllability is expressed in terms of $W(t_o,t_f)$ by the following theorem:

Theorem 3.1 [10]

The system described by Equation (3.1) is completely controllable at time t_o in any control time $t_c (= t_f - t_o)$ if and only if $W(t_o,t_f)$ has rank n for any $t_f > t_o$.

Assuming the system of Equation (3.1) is completely controllable, an optimization problem may now be posed and some necessary conditions for optimality may be presented. This is done in the next section.

3.3 OPTIMIZATION PROBLEM FOR DELAY-FREE SYSTEMS

To facilitate the statement of the optimization problem considered here, and the corresponding necessary conditions for optimality, several definitions will first be made.

Definition 3.4

The performance index (scalar) is given by

$$J = F[x(t_f),t_f] + \int_{t_o}^{t_f} L[x(t),u(t),t]dt \tag{3.3}$$

*$\varphi(\cdot,\cdot)$ is the state transition matrix for both Equation (3.1) and Equation (1.1).

where

t_f is the final time which may or may not be free,

$L[x(t),u(t),t]$ is the cost along the trajectory from the initial time, t_o, to the final time, t_f,

$F[x(t_f),t_f]$ is the cost for being in the state $x(t_f)$ at time t_f.

<u>Definition 3.5</u>

The <u>terminal constraints</u> on the state which the system of Equation (3.1) may have to satisfy are expressed by the set of equations

$$\psi[x(t_f),t_f] = 0 \tag{3.4}$$

where

ψ is a $q \times 1$ vector function, $q \leq n$.

It is assumed that the q constraints in Equation (3.4) are linearly independent. It is necessary, therefore, that $q \leq n$ for the problem to be well posed. Also note that only equality end-point constraints are considered in this discussion. The more general problem with end-point inequality constraints is not treated here, but the theory developed for problems with equality constraints can be easily extended to handle this more general problem [12].

<u>Definition 3.6</u>

The set of all admissible controls, $u(t)$ in Equation (3.1), is designated by U.

It is noted here that the character of the set U distinguishes delay-free from time-delay optimization problems. For the present, however, it is only necessary to state that U must be specified before an explicit optimal control history, $u_{op}(t)$, can be determined.*

The optimal control problem can now be stated precisely in terms of the quantities defined above:

*See [11], for example, on how to explicitly specify the set U.

OPTIMIZATION PROBLEM

Given the system of Equation (3.1) and an initial state x_o, determine the control history $u_{op}(t), t\epsilon[t_o,t_f]$, which minimizes the performance index J in Equation (3.3), satisfies the terminal constraints in Equation (3.4), and which lies in the set of admissible controls, U.

When discussing optimization problems of this type, it is useful to construct the augmented performance index $\bar{J}$ by adjoining the constraint equations (3.4) and the system differential equations (3.1) to the performance index (3.3) with Lagrange multipliers ν and $\lambda(t)$ as follows:

$$\bar{J} = [F[x(t),t] + \nu^T\psi[x(t),t]]_{t=t_f} + \int_{t_o}^{t_f} \Big\{L(x(t),u(t),t) + \lambda^T(t)[A(t)x(t) + B(t)u(t) - \dot{x}(t)]\Big\} dt \tag{3.5}$$

where ν is a $q \times 1$ vector of parameters and $\lambda(t)$ is an $n \times 1$ vector function. If we define the variational Hamiltonian of the system to be

$$H(x,\lambda,u,t) \equiv L(x(t),u(t),t) + \lambda^T(t)[A(t)x(t) + B(t)u(t)],\quad t\epsilon[t_o,t_f], \tag{3.6}$$

then the augmented performance index may be written

$$\bar{J} = [F[x(t),t] + \nu^T\psi[x(t),t]]_{t=t_f} + \int_{t_o}^{t_f} \Big\{H(x,\lambda,u,t) - \lambda^T(t)\dot{x}(t)\Big\} dt. \tag{3.7}$$

Since $\bar{J} = J$ when Equation (3.1) and Equation (3.4) are satisfied, the optimization problem may be reformulated as follows:

REVISED OPTIMIZATION PROBLEM

Given the system of Equation (3.1) and an initial state x_o, determine the control history $u_{op}(t,\lambda,\nu)$, $t\epsilon[t_o,t_f]$, which minimizes the augmented performance index $\bar{J}$ in Equation (3.7) and which lies in the set of admissible controls, U. Choose ν, implicitly, to satisfy the constraint equations (3.4), and choose $\lambda(t)$ such that the system equations (3.1) are satisfied for $t \epsilon [t_o, t_f]$.

The necessary conditions for optimality of $u(t)$ in the revised optimization problem may now be stated as follows:

NECESSARY CONDITIONS FOR OPTIMALITY [12]

It is necessary, if $u(t) = u_{op}(t)$ is optimal in the revised optimization problem, that there exist a vector function $\lambda(t), t \epsilon [t_o, t_f]$, and a vector of parameters ν such that the following conditions are satisfied:

$$\dot{x}(t) = H_\lambda{}^T, \qquad t \epsilon [t_o, t_f] \tag{3.8}$$

$$\dot{\lambda}(t) = -H_x{}^T, \qquad t \epsilon [t_o, t_f] \tag{3.9}$$

$$u_{op}(t) = \underset{u \epsilon U}{\arg\min}\ H(x, \lambda, u, t), \qquad t \epsilon [t_o, t_f] \tag{3.10}$$

$$x(t_o) = x_o \tag{3.11}$$

$$\lambda^T(t_f) = \left(\frac{\partial F}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right)_{t=t_f} \tag{3.12}$$

$$\left[\frac{\partial F}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + \left(\frac{\partial F}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right)(Ax + Bu) + L\right]_{t=t_f} = 0 \tag{3.13}$$

This set of equations, along with the constraint equations (3.4), is mathematically consistent in that it contains: (1) 2n first-order ordinary differential equations (3.8) - (3.9) with 2n boundary conditions (3.11) - (3.12); (2) q parameters ν chosen such that the q algebraic equations (3.4) are satisfied; (3) one parameter t_f to satisfy one algebraic equation (3.13); (4) one vector relationship (3.10) to determine the optimal control vector $\ddot{u}_{op}(t)$.

The condition expressed in Equation (3.13) is referred to as the transversality condition for free end time problems and is used to determine t_f. This equation need not be satisfied, however, when t_f is specified. The calculation of $u_{op}(t)$ from Equation (3.10) is assumed to always be possible in this paper. One is not able to perform the operation indicated in Equation (3.10) explicitly when on a singular arc of the problem*;

*For a discussion of singular arcs, see Reference [12], Chapter 8.

these arcs are not considered in the present analysis. Finally notice that $\lambda(t)$ satisfies the adjoint equations of the system of Equation (3.1), and thus $\lambda(t)$ is termed the adjoint vector of this system.

In the next chapter, the system with time delay, analogous to that of Equation (3.1), will be discussed. The results of the present chapter will be used in the analysis of this system.

CHAPTER IV

OPTIMAL CONTROL OF LINEAR SYSTEMS POSSESSING TIME DELAYS

4.1 SYSTEM DESCRIPTION

The system of interest in this chapter possesses dynamics described by Equation (1.1). This system is completely specified by these dynamical equations, by the initial state of the system, and by the initial state of the delay (see Section 2.1). For convenience, these quantities are listed below:

$$\dot{x}(t) = A(t)x(t) + B(t)\, u(t - t_d) \tag{4.1}$$

$$x(t_o) = x_o \tag{4.2}$$

$$u(t - t_d) = u_o(t), \qquad t_o \le t < t_o + t_d \tag{4.3}$$

It is assumed that t_d has a finite magnitude, and that the specified vector function $u_o(t)$ is a member of the set U of admissible control functions (Definition 3.6). Except for this restriction, $u_o(t)$ is an arbitrary, specified vector function.

Referring to Equation (2.1) and Equation (4.3), the solution of Equation (4.1) for $x(t)$ may be written

$$x(t) = \varphi(t,t_o)x_o + \int_{t_o}^{t_o + t_d} \varphi(t,\tau)B(\tau)u_o(\tau)d\tau$$

$$+ \int_{t_o + t_d}^{t} \varphi(t,\tau)B(\tau)u(\tau - t_d)d\tau, \qquad t \ge t_o + t_d \tag{4.4}$$

where, again, $\varphi(\cdot,\cdot)$ is the state transition matrix of both Equation (4.1) and Equation (3.1). It is thus seen that the state of the system, $x(t)$, at time $t > t_o + t_d$, is a function of the initial state x_o and a functional of the initial state of the delay $u_o(t)$, $t \epsilon [t_o, t_o + t_d]$.

A discussion on controllability of Equations (4.1) - (4.3) is presented in the next section, paralleling the one given for delay-free systems in Section 3.2.

4.2 CONTROLLABILITY OF SYSTEMS WITH TIME DELAY

Definition 3.3 for system controllability applies equally well to systems with, or without, time delay in the control. The concepts of "state transfer" and of "control time" (see Definitions (3.1) - (3.2)) will have significance in the present discussion, despite their irrelevance in Chapter III.

First define the "controllability matrix" for the system characterized by Equation (4.1) to be

$$W(t_o + t_d, t_f) \equiv \int_{t_o + t_d}^{t_f} \varphi(t_o + t_d, \tau)B(\tau)B^T(\tau)\varphi^T(t_o + t_d, \tau)d\tau \tag{4.5}$$

The following theorem gives a necessary and sufficient condition, in terms of $t_c (= t_f - t_o)$ and $W(t_o + t_d, t_f)$, for insuring the controllability of a state x_o at time t_o associated with the system of Equations (4.1) - (4.3).

Theorem 4.1

A state $x(t_o) = x_o$, associated with the system of Equations (4.1) - (4.3), is controllable at time t_o in control time t_c if and only if (1) $W(t_o + t_d, t_f)$ has rank n for any $t_f > t_o + t_d$, and (2) $t_c > t_d$.*

Proof

Assume $t_c > t_d$. Under this assumption it will first be shown that the existence of $W^{-1}(t_o + t_d, t_f)$ is sufficient for controllability of x_o.

*Condition (2) of this theorem implies that the state $x(t)$ is uncontrollable when $t \in [t_o, t_o + t_d]$. Hence, from Definition (3.3), any system with a time delay in the control cannot be completely controllable.

Suppose the theorem is true. Then, according to Definition (3.3), there exists a time $t_f > t_o + t_d$ and a control function $u(t - t_d)$ such that $x(t_f) = 0$ for all $t \geq t_f$. Since $t_c > t_d$, Equation (4.4) is valid at $t = t_f$. Evaluating Equation (4.4) at $t = t_o + t_d$ to obtain $x(t_o + t_d)$, and then solving for $x(t_f)$ in terms of $x(t_o + t_d)$ gives

$$x(t_o + t_d) = \varphi(t_o + t_d, t_o)x_o + \int_{t_o}^{t_o + t_d} \varphi(t_o + t_d, \tau)B(\tau)u_o(\tau)d\tau \tag{4.6}$$

$$x(t_f) = \varphi(t_f, t_o + t_d)x(t_o + t_d) + \int_{t_o + t_d}^{t_f} \varphi(t_f, \tau)B(\tau)u(\tau - t_d)d\tau \tag{4.7}$$

According to the theorem, $W^{-1}(t_o + t_d, t_f)$ exists for all $t_f > t_o + t_d$. Thus the following control function is a candidate for $u(\tau - t_d)$, $\tau\epsilon[t_o + t_d, t_f]$:

$$u(\tau - t_d) = -B^T(\tau)\varphi^T(t_o + t_d, \tau)W^{-1}(t_o + t_d, t_f)x(t_o + t_d), \tau\epsilon[t_o + t_d, t_f] \tag{4.8}$$

It will be shown in Chapter V that this control function minimizes the performance criterion

$$J = \frac{1}{2} \int_{t_o}^{t_f} u^T(\tau - t_d)u(\tau - t_d)d\tau, \qquad t_f > t_o + t_d,$$

with $x(t_f)$ constrained to be zero. This fact is incidental, however, since any control law which contains $W^{-1}(t_o + t_d, t_f)$ and makes $x(t_f) = 0$ is all that is needed here. Substitution of Equation (4.8) into Equation (4.7) gives $x(t_f) = 0$. Then, from Equation (4.1), setting $u(\tau - t_d) = 0$, $\tau > t_f$, results in $x(t) = 0$, $t \geq t_f$. Thus, the existence of $W^{-1}(t_o + t_d, t_f)$ when $t_c > t_d$ is sufficient to insure controllability of the initial state x_o.

To prove necessity when $t_c > t_d$, assume (1) $W(t_o + t_d, \bar{t}_f)$ is singular for some final time $\bar{t}_f$, and (2) the state associated with the system of Equations (4.1) - (4.3) is controllable at time t_o. Then, there exists a control vector function $\bar{u}_o(\tau), \tau\epsilon[t_o, t_o + t_d]$, which gives a non-zero state $\bar{x}(t_o + t_d)$ at time $t = t_o + t_d$ (see Equation (4.6)) having the following property*:

$$W(t_o + t_d, \bar{t}_f)\bar{x}(t_o + t_d) = 0 \tag{4.9}$$

The existence of $\bar{u}_o(\tau), \tau\epsilon[t_o, t_o + t_d]$ follows from the fact that $u_o(\tau)$ is an arbitrary function and $W(t_o + t_d, \bar{t}_f)$ is singular. Now, since the state x_o is controllable by Assumption (2), there exists a control function $\bar{u}(\tau - t_d)$, $\tau\epsilon[t_o + t_d, t_f]$, such that $x(t_f) = 0$ in Equation (4.7). From this equation we obtain

$$\bar{x}(t_o + t_d) = -\int_{t_o + t_d}^{\bar{t}_f} \varphi(t_o + t_d, \tau)B(\tau)\bar{u}(\tau - t_d)d\tau \tag{4.10}$$

From Equation (4.9), $\bar{x}^T(t_o + t_d)W(t_o + t_d, \bar{t}_f)\bar{x}(t_o + t_d) = 0$. Using Equation (4.5) in this expression then gives

$$\int_{t_o + t_d}^{\bar{t}_f} [B^T(\tau)\varphi^T(t_o + t_d, \tau)\bar{x}(t_o + t_d)]^T[B^T(\tau)\varphi^T(t_o + t_d, \tau)\bar{x}(t_o + t_d)]d\tau = 0 \tag{4.11}$$

Equation (4.11) implies

$$B^T(\tau)\varphi^T(t_o + t_d, \tau)\bar{x}(t_o + t_d) = 0, \ \tau\epsilon[t_o + t_d, \bar{t}_f] \tag{4.12}$$

Multiplying Equation (4.10) by $\bar{x}^T(t_o + t_d)$ and then using Equation (4.12)

*If $\bar{x}(t_o + t_d) = 0$, then, since the origin is an equilibrium point of the uncontrolled system of Equation (4.1), $u(\tau - t_d) = 0$, $\tau \geq t_o + t_d$, insures that $x(t) = 0$, $t \geq t_o + t_d$. Thus, $\bar{x}(t_o + t_d)$ is assumed to be non-zero since we are interested in determining whether $\bar{x}(t_o + t_d) \neq 0$ can be transferred to the origin in control time $t_c > t_d$ while satisfying Assumptions (1) - (2) above.

finally gives

$$\bar{x}^T(t_o + t_d)\bar{x}(t_o + t_d) = -\int_{t_o + t_d}^{\bar{t}_f} \bar{x}^T(t_o + t_d)\varphi(t_o + t_d,\tau) \cdot B(\tau)\bar{u}(\tau - t_d)d\tau = 0 \tag{4.13}$$

Equation (4.13) implies $\bar{x}(t_o + t_d) = 0$ which is a contradiction. Therefore it is necessary, as well as sufficient, that $W^{-1}(t_o + t_d,t_f)$ exists when $t_c > t_d$.

It remains to show that if $t_c < t_d$, then the state x_o is not controllable.* When $t_f < t_o + t_d$, the solution of Equations (4.1) - (4.3) for $x(t_f)$ is

$$x(t_f) = \varphi(t_f,t_o)x_o + \int_{t_o}^{t_f} \varphi(t_f,\tau)B(\tau)u_o(\tau)d\tau, \qquad t_f < t_o + t_d \tag{4.14}$$

Recall that $u_o(\tau)$, $\tau\epsilon[t_o,t_o + t_d]$, is arbitrary and prescribed. Since $x(t_f)$ in Equation (4.14) is independent of $u(\cdot)$, there is no way to place $x(t_f) \neq 0$ at the origin once x_o and $u_o(\tau)$, $\tau\epsilon[t_o,t_o + t_d]$, are prescribed. If, by chance, $x(t_f) = 0$, then it is necessary that $u_o(\tau) = 0$, $\tau\epsilon[t_f,t_o + t_d]$, if the state is to remain at zero for $t \geq t_f$. In general, this will not be true. Thus it is concluded that the state associated with the system of Equations (4.1) - (4.3) can be controllable only if $t_c \geq t_d$. This completes the proof of the theorem.

It is of interest to compare the controllability requirements for the delay-free system (Theroem 3.1) and the analogous time-delay system (Theorem 4.1). When $t_f < t_o + t_d$, the time-delay system is uncontrollable, regardless of the controllability of the delay-free system. Therefore assume $t_f > t_o + t_d$ and, for purposes of comparison, assume t_f is fixed

*The uninteresting case of $t_c = t_d$ is discussed in the footnote on page 35.

and t_o is free. Now note that the integrand of the controllability matrix for both systems is positive semi-definite. Define the integrand of the controllability matrix for the time-delay system to be

$$I(t_o + t_d,\tau) \equiv [\varphi(t_o + t_d,\tau)B(\tau)][\varphi(t_o + t_d,\tau)B(\tau)]^T, \tau\epsilon[t_o + t_d,t_f] \tag{4.15}$$

The integrand of the controllability matrix for the delay-free system, $I(t_o,\tau)$, may then be written

$$I(t_o,\tau) = \begin{cases} \varphi(t_o,t_o + t_d)I(t_o + t_d,\tau)\varphi^T(t_o,t_o + t_d), & \tau\epsilon[t_o + t_d,t_f] \\ [\varphi(t_o,\tau)B(\tau)][\varphi(t_o,\tau)B(\tau)]^T & , \tau\epsilon[t_o,t_o + t_d] \end{cases} \tag{4.16}$$

From Theorem 4.1, if the time-delay system is controllable for some t_o, then $I(t_o + t_d,\tau)$ is non-zero over some finite interval on the τ-axis, $\tau\epsilon[t_o + t_d,t_f]$. Then Equation (4.16) implies that $I(t_o,\tau)$ is non-zero over the same interval on the τ-axis and hence, from Theorem 3.1, the delay-free system is controllable at time t_o. The converse is not true since controllability of the delay-free system at time t_o implies $I(t_o,\tau)$ is non-zero over some finite interval on the τ-axis, $\tau\epsilon[t_o,t_f]$. This non-zero segment on the τ-axis may, however, occur in the interval $[t_o,t_o + t_d]$, which, from Equation (4.16), implies nothing about $I(t_o + t_d,\tau)$, $\tau\epsilon[t_o + t_d,t_f]$.

To summarize the above discussion, the following conclusions have been drawn here:

1) For a given $t_f > t_o + t_d$, the controllability of a state $x(t_o) = x_o$, associated with the time-delay system, implies controllability of this state in the analogous delay-free system.

2) For a given $t_f > t_o + t_d$, the controllability of a state $x(t_o) = x_o$, associated with the delay-free system, implies nothing about the controllability of this state in the analogous time-delay system.

4.3 OPTIMIZATION PROBLEM FOR SYSTEMS WITH TIME DELAY

In this section an optimization problem, along with the corresponding necessary conditions for optimality, is formulated for linear systems with time delay in the control. Both the system of interest and the optimization problem posed below are analogous to those of Section (3.3). The notation of that section will be retained wherever possible. In fact, the optimization problem for systems with time delay will be posed so that the results of Section (3.3) can be used directly, to determine the necessary conditions for optimality.

First, define a new control function, $v(t)$, such that the system equations may be written

$$\dot{x}(t) = A(t)x(t) + B(t)v(t) \tag{4.17}$$

$$x(t_o) = x_o \tag{4.18}$$

where

$$v(t) \equiv \begin{Bmatrix} u_o(t), & t_o \leq t < t_o + t_d \\ u(t - t_d), & t_o + t_d \leq t \leq t_f \end{Bmatrix} \tag{4.19}$$

Thus $v(t)$, an $m \times 1$ vector function, is the control being sought in a well posed optimization problem. Note that, in general, $v(t)$ will be discontinuous at $t = t_d + t_o$. Also recall, from Section (4.2), that a state of the system of Equations (4.17) - (4.19) is controllable only if $t_f \geq t_d + t_o$. It is hereafter assumed, therefore, that $t_f \geq t_d + t_o$.

The use of $v(t)$ in Equation (4.17) does not eliminate any problems created by delays in the system differential equations. It does, however, make explicit the basic difference between optimization problems for systems with, and without, time delays in the control. The following definition should make precise the basic distinction between the two systems:

Definition 4.1

A control function $v(t)$, $t \epsilon [t_o, t_f]$, is a member of the set of admissible control functions, V, for systems with time delay, if and only if $v(t)$ satisfies the following conditions:

(i) $v(t) = u_o(t), \quad t\epsilon[t_o,t_o + t_d]$

(ii) $u_o(t)\epsilon U, \quad t\epsilon[t_o,t_o + t_d]$

(iii) $v(t)\epsilon U, \quad t\epsilon[t_o + t_d,t_f]$

where U is the set of admissible controls for the analogous delay-free system and $u_o(t)$ is a prescribed vector function of time.

Hence, comparing Equation (4.17) with Equation (3.1), it is seen that the time-delay system is identical to the delay-free system, excpet for the difference in the sets of admissible controls, V and U, from which the control function for each system must be chosen. Thus, if U is replaced by V, and $u(t)$ by $v(t)$, in Section (3.3), the results of that section for delay-free systems become valid for systems with time delay. This is precisely what has been done below. The numbers in parentheses above the equality signs in the following equations refer to the analogous (and sometimes identical) equations for delay-free systems.

The performance index may be written

$$J' \overset{(3.3)}{=} F[x(t_f),t_f] + \int_{t_o}^{t_f} L[x(t),v(t),t]dt. \qquad (4.20)$$

From Equation (4.4), $x(t)$ is a functional of $u_o(t)$ for $t\epsilon[t_o,t_o + t_d]$. Thus, during this time interval, since J' is not influenced by the, as yet, undetermined control function, the performance index may be redefined to be

$$J = F[x(t_f),t_f] + \int_{t_o + t_d}^{t_f} L[x(t),v(t),t]dt. \qquad (4.21)$$

Thus one is now interested in minimizing J over the time interval $[t_o + t_d,t_f]$ and such that the end-point constraints

$$\psi[x(t_f),t_f] \overset{(3.4)}{=} 0 \qquad (4.22)$$

are satisfied.

Using the same argument which allowed us to define J by Equation (4.21), the variational Hamiltonian and augmented performance index for the system may be written

$$H(x,\lambda,v,t) \overset{(3.6)}{\equiv} L(x(t),v(t),t) + \lambda^T(t)[A(t)x(t) + B(t)v(t)], \quad t\epsilon[t_o + t_d, t_f] \tag{4.23}$$

$$\bar{J} \overset{(3.7)}{\equiv} [F[x(t),t] + \nu^T\psi[x(t),t]]_{t = t_f} + \int_{t_o + t_d}^{t_f} \left\{ H(x,\lambda,v,t) - \lambda^T(t)\dot{x}(t) \right\} dt. \tag{4.24}$$

Now the optimization problem may be formally stated as follows:

OPTIMIZATION PROBLEM

Given the system of Equation (4.17) and an initial state x_o, determine the control history $v_{op}(t,\lambda,\nu), t\epsilon[t_o + t_d, t_f]$, which minimizes the augmented performance index $\bar{J}$ in Equation (4.24) and which lies in the set of admissible controls V. Choose ν, implicitly, to satisfy the constraint equations (4.22), and choose $\lambda(t)$ such that the system equations (4.17) are satisfied for $t\epsilon[t_o + t_d, t_f]$.

Now notice that $x(t)$ occurs in the above expressions only when $t \geq t_o + t_d$. Assuming no disturbances act on the system in the time interval $[t_o, t_o + t_d]$, the state at $t = t_o + t_d$ is given in Equation (4.6). Since x_o and $u_o(t), t\epsilon[t_o, t_o + t_d]$, are prescribed, $x(t_o + t_d)$ is prescribed. Also note that the knowledge of x_o and $u_o(t), t\epsilon[t_o, t_o + t_d]$, is necessary only for the purpose of calculating $x(t_o + t_d)$. From Definition (4.1) it is seen that the choice of the control law, $v(t)$, $t\epsilon[t_o + t_d, t_f]$, is actually made in the set U of admissible controls for the delay-free system. The optimization problem may thus be reformulated:

REVISED OPTIMIZATION PROBLEM

Given the system of Equation (4.17) and the state $x(t_o + t_d)$, determine the control history $v_{op}(t,\lambda,\nu), t\epsilon[t_o + t_d, t_f]$, which minimizes the augmented performance index $\bar{J}$ in Equation (4.24) and which lies in the set of admissible controls U. Choose ν, implicitly, to satisfy the constraint equations (4.22), and choose $\lambda(t)$ such that the system equations (4.17) are satisfied for $t \epsilon [t_o + t_d, t_f]$.

The necessary conditions for optimality of $v(t)$ in the revised optimization problem are now presented. The revised optimization problem in this section is identical to that of Section (3.3) for the delay-free system when t_o is replaced by $t_o + t_d$. Thus the necessary conditions for optimality, Equations (3.8) - (3.13), are identical for the two systems when this association is made. These conditions are repeated here for the time-delay system. A direct proof, from the calculus of variations viewpoint, is given in Appendix A.

NECESSARY CONDITIONS FOR OPTIMALITY

It is necessary, if $v(t) = v_{op}(t)$ is optimal in the revised optimization problem, that there exist a vector function $\lambda(t), t \epsilon [t_o + t_d, t_f]$, and a vector of parameters ν such that the following conditions are satisfied:

$$\dot{x}(t) = H_\lambda^T, \quad t \epsilon [t_o + t_d, t_f] \tag{4.25}$$

$$\dot{\lambda}(t) = -H_x^T, \quad t \epsilon [t_o + t_d, t_f] \tag{4.26}$$

$$v_{op}(t) = \arg \min_{v \epsilon U} H(x, \lambda, v, t), \quad t \epsilon [t_o + t_d, t_f] \tag{4.27}$$

$$x(t_o + t_d) \text{ specified} \tag{4.28}$$

$$\lambda^T(t_f) = \left(\frac{\partial F}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right)_{t = t_f} \tag{4.29}$$

$$\left[\frac{\partial F}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + \left(\frac{\partial F}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right)(Ax + Bv) + L\right]_{t = t_f} = 0 \tag{4.30}$$

The discussion following the statement of the necessary conditions for optimality of delay-free systems is also applicable here. Comparing Equations (4.25) - (4.30) with Equations (3.8) - (3.13) reveals the following basic fact, stated here in the form of a theorem:

Theorem 4.2

The necessary conditions for optimality of delay-free linear systems

(Equation (3.1)) are identical to the necessary conditions for optimality of analogous systems with time delay in the control (Equations (4.1) - (4.3)) if

(1) both systems possess the same state at time $t = t_o + t_d$,

(2) $t \geq t_o + t_d$.

The consequences of this result, with regard to control of linear systems with time delay in the control, is discussed in the next chapter. It should be apparent now that the ability with which one can optimally control systems with time delay in the control input is limited to ones ability of optimally controlling the analogous delay-free systems.

CHAPTER V

CALCULATION AND IMPLEMENTATION OF OPTIMAL CONTROL LAWS

5.1 RELATIONSHIP BETWEEN OPTIMAL CONTROL LAWS FOR DELAY-FREE AND TIME-DELAY SYSTEMS

Assume that the optimal control law for the delay-free system has been determined by satisfying the necessary conditions for optimality (Equation (3.8)-Equation (3.13)) of the optimization problem posed in Section 3.3. This control law will, in general, be a function of the state, $x(t)$, as well as time. Assuming that the control law is a feedback solution of the optimization problem, $u_{op}(t)$ may be written

$$u_{op}(t) = k(x(t),t) \tag{5.1}$$

The difficulty in obtaining $k(x(t),t)$ for a general performance index and a high-order system is most often appreciable. The intent of this section, however, is to show that once $k(x(t),t)$ is known, then the optimal control law for the analogous time-delay optimization problem is also known.

Theorem 4.2 relates the necessary conditions for optimality between the delay-free and the analogous time-delay optimization problems. Starting at time $t = t_o + t_d$ and state $x(t_o + t_d)$, the optimal trajectories for each system are identical since, by Theorem 4.2, the necessary conditions for optimality are identical. Now notice, from Figure 1.2, that the control signal at time t is generated a time t_d prior to its execution. If the trajectories of the time-delay and delay-free systems are to match after time $t = t_o + t_d$, it is necessary that the calculation of the control at time t be based upon knowledge of the state at time $t + t_d$. Figure 5.1 shows diagramatically how the optimal control law for the time-delay system can be calculated from the optimal control law for the delay-free system by predicting the value of the state at time $t + t_d$. Figure 5.2 shows a typical control history for each system which results when using this control scheme.

The degree of success in implementing this control law is directly related to the ability of predicting the state at time $t + t_d$, knowing only the present state of the system and the present state of the delay.

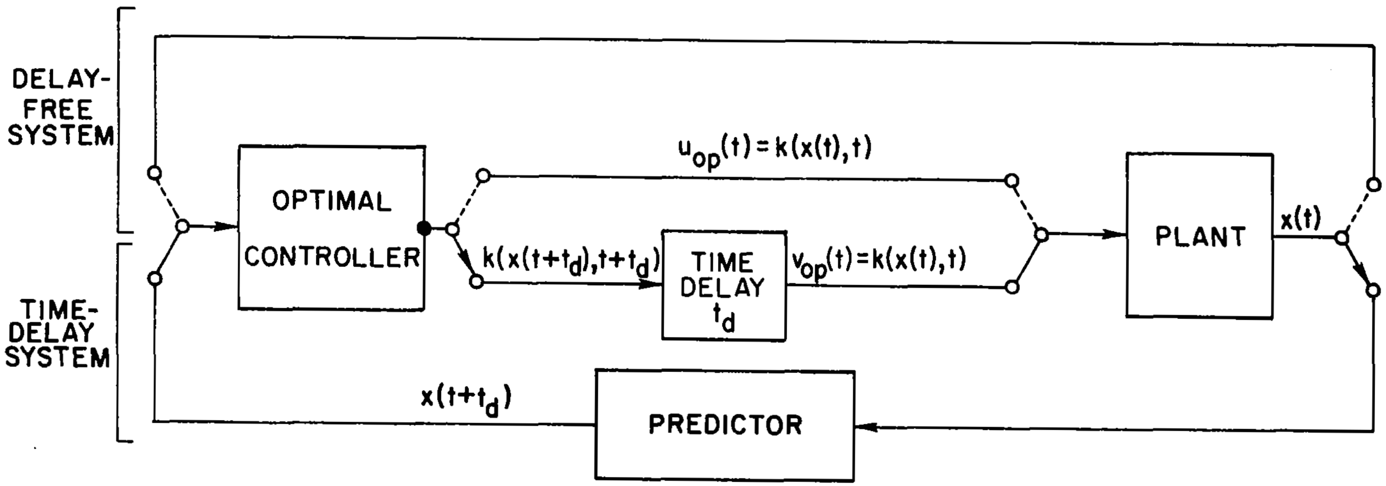


Figure 5.1. Relationship Between Optimal Controllers for Delay-Free and Time-Delay Systems When $t \geq t_o + t_d$.

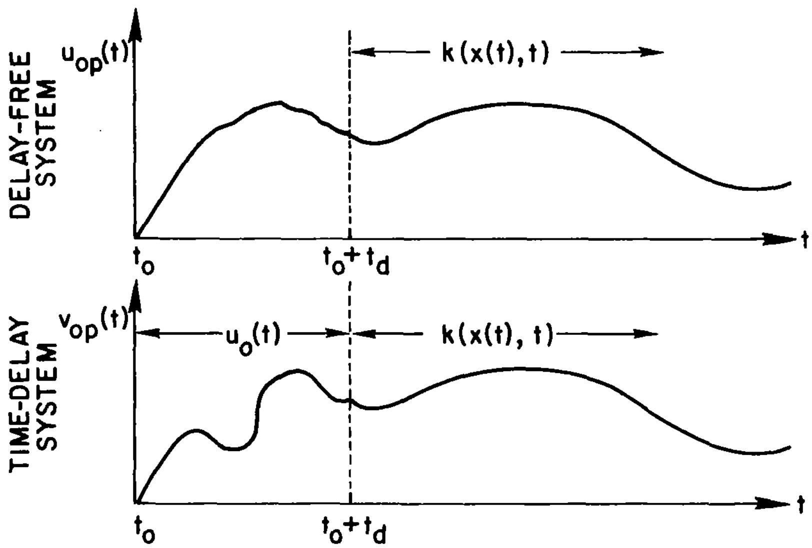


Figure 5.2. Typical Control Histories for Delay-Free and Time-Delay Systems.

If the time delay is very large and if the disturbances acting on the plant are basically unpredictable, the resulting trajectories would be sub-optimal due to the inability to accurately predict the state. These trajectories are not optimal during the time interval, of magnitude t_d, between the initiation of an unpredicted disturbance and the time at which the resulting control signal first responds to this disturbance.

Some example problems are discussed in the next two sections to illustrate the ideas presented above. In Section 5.2 the optimal control law for an n^{th}-order system with time delay and with unbounded control is derived analytically for a quadratic performance index. The second-order example of Section 2.2.2 is reconsidered in Section 5.3.

5.2 OPTIMAL CONTROL OF n^{th}-ORDER SYSTEMS WITH UNBOUNDED CONTROL AND QUADRATIC PERFORMANCE INDEX

Consideration of optimal control problems with unbounded control is motivated primarily by the ease with which the Hamiltonian of the system may be minimized (Equation (4.27)) when no bound is placed on the control. This operation is particularly simple when the system is linear and the performance index is quadratic in the control. Two such problems are considered in this section, one fixed end-point problem and one free end-point problem. The necessary conditions for optimality (Equations (4.25) - (4.30)) will be used directly in obtaining solutions to these problems.

5.2.1 FIXED END-POINT PROBLEM WITH SPECIFIED FINAL TIME

The system of Equations (4.1) - (4.3) is considered here. The system is assumed to be controllable for $t_f > t_o + t_d$, where t_f is the specified final time. Define the performance index for this problem to be

$$J = \frac{1}{2}\int_{t_o + t_d}^{t_f} u^T(\tau - t_d)Q_2 u(\tau - t_d)d\tau \qquad (5.2)$$

where Q_2 is a symmetric, non-singular $m \times m$ weighting matrix.* The

*A weighting matrix is also assumed to be positive definite in order to give meaningful results.

control problem is to transfer the initial state to the origin while minimizing J. The terminal constraints are thus specified by

$$\psi[x(t_f),t_f] = x(t_f) = 0. \tag{5.3}$$

The class, U, of admissible controls is assumed to be broad enough to make the calculations below valid. In particular, it is assumed that $u_i(\tau - t_d)$, the components of $u(\tau - t_d)$, may be unbounded so that minimization of $H(x,\lambda,v,t)$ may be accomplished by setting $H_v = 0$ and solving for $v(t), t\epsilon[t_o + t_d, t_f]$.*

The calculation of $v_{op}(t)$ is thus accomplished as follows:

The variational Hamiltonian, from Equation (4.23), is

$$H(x,\lambda,v,t) = \frac{1}{2} v^T Q_2 v + \lambda^T[Ax + Bv], \; t\epsilon[t_o + t_d, t_f] \tag{5.4}$$

and the optimal control law is obtained by using Equation (4.27):

$$v_{op}(t,\lambda,\nu) = \underset{v\epsilon U}{\arg\min}\; H(x,\lambda,v,t) = -Q_2^{-1}B^T\lambda, \; t\epsilon[t_o + t_d, t_f]. \tag{5.5}$$

From Equation (4.26) and Equation (4.29), the adjoint variables are solutions of

$$\dot{\lambda}(t) = -A^T(t)\lambda(t), \; \lambda(t_f) = \nu, \quad t\epsilon[t_o + t_d, t_f]. \tag{5.6}$$

Integrating Equation (5.6) backwards gives

$$\lambda(t) = \boldsymbol{\varphi}^T(t_f,t)\nu, \; t\epsilon[t_o + t_d, t_f] \tag{5.7}$$

where $\boldsymbol{\varphi}(\cdot,\cdot)$ is the state transition matrix of Equation (4.1). Recall that since $x(t_o + t_d)$ can be calculated from Equation (4.6), it is considered specified. Thus, integrating Equation (4.1) forward, using Equation (5.5) and Equation (5.7), gives

$$x(t) = \varphi(t,t_o + t_d)x(t_o + t_d) - \left[\int_{t_o + t_d}^{t} \varphi(t,\tau)B(\tau)Q_2^{-1}B^T(\tau)\varphi^T(t_f,\tau)d\tau\right]\nu. \tag{5.8}$$

*More precisely, $H_v = 0$ determines an extremum of J. Further analysis is required to show that J is minimized.

Since ν is chosen to satisfy the terminal constraints, ν is found from Equation (5.8) with $t = t_f$. By utilizing the properties of $\phi(\cdot,\cdot)$, ν may be written in the form

$$\nu = \varphi^T(t_o + t_d, t_f)\left[\int_{t_o + t_d}^{t_f} \varphi(t_o + t_d, \tau)B(\tau)Q_2^{-1}B^T(\tau)\varphi^T(t_o + t_d, \tau)d\tau\right]^{-1} \cdot x(t_o + t_d) \tag{5.9}$$

where the existence of the inverse is insured by the non-singularity of Q_2 and by the controllability assumption (existence of $W^{-1}(t_o + t_d, t_f)$). Substituting Equation (5.9) and Equation (5.7) into Equation (5.5) finally gives the optimal control law for this problem:

$$v_{op}(t) = -Q_2^{-1}B^T(t)\varphi^T(t_o + t_d, t)\left[\int_{t_o + t_d}^{t_f} \varphi(t_o + t_d, \tau)B(\tau)Q_2^{-1}B^T(\tau) \cdot \varphi^T(t_o + t_d, \tau)d\tau\right]^{-1} x(t_o + t_d),\ t \in [t_o + t_d, t_f]. \tag{5.10}$$

This control drives the initial state to zero at time t_f and the control $v(t) = 0$, $t \geq t_f$, holds the state at the origin. This completes the solution of this optimization problem.

Setting $Q_2 = I_m$, the $m \times m$ identity matrix, in Equation (5.10), $v_{op}(t)$ reduces to the control law (Equation (4.8)) derived in Section 4.2 for zeroing the state. Also note that Equation (5.10) gives the sampled-data* open loop solution to the optimization problem.

The continuous feedback form of the solution to the optimization problem is found by replacing t_o with t in the above discussion. It was assumed that $t_o + t_d < t_f$ and hence the continuous feedback solution can only be valid when $t < t_f - t_d$. This is expected since after time $t_f - t_d$, $t_c < t_d$ and, from Theorem 4.1, the system is no longer controllable. After time $t_f - t_d$, therefore, the sampled-data version of the solution must be used.

Replacing t_o by t in Equation (5.10), the feedback solution is

*The solution is sampled-data in the limited sense where the number of samples is one, namely the sample x_o at time t_o.

$$v_{op}(t) = -Q_2^{-1}B^T(t)\varphi^T(t + t_d,t)[\int_{t + t_d}^{t_f} \varphi(t + t_d,\tau)B(\tau)Q_2^{-1}B^T(\tau)\cdot \tag{5.11}$$
$$\varphi^T(t + t_d,\tau)d\tau]^{-1}x(t + t_d), t < t_f - t_d$$

where $x(t + t_d)$ may be written

$$x(t + t_d) = \varphi(t + t_d,t)x(t) + \int_t^{t + t_d} \varphi(t + t_d,\tau)B(\tau)v_{op}(\tau)d\tau. \tag{5.12}$$

Thus, from Equation (5.12), the optimal feedback control law is a linear function of the present state of the system and a functional of the present state of the delay. This latter fact tends to complicate the implementation of this time-variable feedback control law.

Finally note that the control gain in Equation (5.11) approaches infinity as $t \to t_f - t_d$. This control behavior is expected whenever terminal constraints are required to be satisfied exactly. To alleviate this problem, these terminal constraints are relaxed in the next problem and, instead, are replaced by a quadratic term in the final state in the performance index.

5.2.2 FREE END-POINT PROBLEM WITH SPECIFIED FINAL TIME

The problem considered here is identical to the problem above with the exception that the terminal constraint, $\psi = 0$, is removed, and the performance index is modified to be

$$J = \frac{1}{2} x^T(t_f)Q_1x(t_f) + \frac{1}{2}\int_{t_o + t_d}^{t_f} u^T(\tau - t_d)Q_2u(\tau - t_d)d\tau \tag{5.13}$$

where Q_1 is a symmetric, non-singular $n \times n$ weighting matrix. The calculation of $v_{op}(t)$ for this problem is similar to that of the fixed end-point problem. This calculation is sketched below.

The variational Hamiltonian is given by

$$H(x,\lambda,v,t) = \frac{1}{2} v^TQ_2v + \lambda^T[Ax + Bv], \quad t\epsilon[t_o + t_d,t_f] \tag{5.14}$$

and $v_{op}(t,\lambda)$ is written

$$v_{op}(t,\lambda) = -Q_2^{-1}B^T(t)\lambda(t), \quad t\epsilon[t_o + t_d, t_f]. \tag{5.15}$$

The divergence from the solution of the fixed end-point problem occurs in the solution for the adjoint variables:

$$\dot{\lambda}(t) = -A^T\lambda(t), \lambda(t_f) = Q_1x(t_f), \quad t\epsilon[t_o + t_d, t_f] \tag{5.16}$$

$$\lambda(t) = \varphi^T(t_f,t)Q_1x(t_f), \quad t\epsilon[t_o + t_d, t_f]. \tag{5.17}$$

The final state, using this control law, may be written

$$x(t_f) = Q_1^{-1}[Q_1^{-1} + \varphi(t_f,t_o + t_d)[\int_{t_o + t_d}^{t_f} \varphi(t_o + t_d,\tau)B(\tau)Q_2^{-1}B^T(\tau) \cdot$$
$$\varphi^T(t_o + t_d,\tau)d\tau]\varphi^T(t_f,t_o + t_d)]^{-1}\varphi(t_f,t_o + t_d)x(t_o + t_d) \tag{5.18}$$

where the inverse exists since Q_1^{-1} is assumed non-singular and the integral term is positive semi-definite. Finally, substitute Equation(5.18) and Equation (5.17) into Equation (5.15) to get $v_{op}(t)$:

$$v_{op}(t) = -Q_2^{-1}B^T(t)\varphi^T(t_f,t)[Q_1^{-1} + \varphi(t_f,t_o + t_d)[\int_{t_o + t_d}^{t_f} \varphi(t_o + t_d,\tau) \cdot$$
$$B(\tau)Q_2^{-1}B^T(\tau)\varphi^T(t_o + t_d,\tau)d\tau]\varphi^T(t_f,t_o + t_d)]^{-1}\varphi(t_f,t_o + t_d)x(t_o + t_d),$$
$$t\epsilon[t_o + t_d, t_f] \tag{5.19}$$

Now, if $v(t) = 0$, $t \geq t_f$, and if the origin is at least stable within a region containing $x(t_f)$, then the state will remain in some neighborhood of the origin, as desired. This completes the solution of this optimization problem.

Note that, again, Equation (5.19) gives the sampled-data, open loop solution to the optimization problem. The continuous feedback control law is obtained from Equation (5.19) by replacing t_o with t. Again, this feedback solution is valid only when $t \leq t_f - t_d$. The feedback solution is written

$$v_{op}(t) = -Q_2^{-1} B^T(t)\varphi^T(t_f,t)[Q_1^{-1} + \varphi(t_f,t_o + t_d)[\int_{t+t_d}^{t_f} \varphi(t + t_d,\tau)\cdot$$

$$B(\tau)Q_2^{-1}B^T(\tau)\varphi^T(t + t_d,\tau)d\tau]\varphi^T(t_f,t + t_d)]^{-1}\varphi(t_f,t+t_d)x(t+t_d), \qquad (5.20)$$

$$t \le t_f - t_d$$

Again, $v_{op}(t)$ is a function of the present state of the system and a functional of the present state of the delay; but note that $v_{op}(t)$ is finite in magnitude for all $t \le t_f - t_d$ and for any t_f, as long as $x(t)$ is finite for all time. A finite gain controller has thus been obtained by accepting a non-zero final state. The proper choices for Q_1, Q_2, and t_f in Equation (5.18) can, however, make $x(t_f)$ arbitrarily small and still bound the magnitude of the control effort.

The above two problems were solved by satisfying the necessary conditions for optimality directly. This was possible since the adjoint equations were easily integrated and the minimization of the Hamiltonion easily yielded the control law in terms of the adjoint variables. The problem considered in the next section is not easily solved by using the necessary conditions directly. Instead, the ideas discussed in Section 5.1 are utilized to obtain a feedback control law for the optimization problem.

5.3 SECOND-ORDER EXAMPLE WITH BOUNDED CONTROL AND FREE FINAL TIME

The problem considered in this section is the minimum-time regulator problem for the 2^{nd} -order plant of Section 2.2. The system dynamics and problem statement are summarized here for convenience:

$$\begin{aligned} \ddot{e}(t) + 2\zeta\dot{e}(t) + e(t) &= u(t - t_d), \qquad t\epsilon[t_o + t_d,t_f] \\ &= 0, \qquad t\epsilon[t_o,t_o + t_d] \end{aligned} \qquad (5.21)$$

$$|u(t - t_d)| \le N, \qquad t\epsilon[t_o + t_d,t_f] \qquad (5.22)$$

$$e(t_o) = e_o,\dot{e}(t_o) = \dot{e}_o;\ e(t) = \dot{e}(t) = 0, \quad t < t_o \qquad (5.23)$$

$$\psi[x(t_f),t_f] = x(t_f) = 0, \ t_f \text{ free} \tag{5.24}$$

$$J = \int_{t_o + t_d}^{t_f} dt = t_f - (t_o + t_d) \tag{5.25}$$

where the state $x(t)$ is given by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix}. \tag{5.26}$$

The adjoint vector is written $\lambda^T = [\lambda_1, \lambda_2]$. The variational Hamiltonian for this system is thus

$$H(x,\lambda,v,t) = 1.0 + \lambda_1 x_2 + \lambda_2[-x_1 - 2\zeta x_2 + v] \tag{5.27}$$

The necessary conditions for optimality (Equations (4.25) - (4.30)) reduce to

$$\begin{aligned} \dot{\lambda}_1(t) &= \lambda_2(t), \qquad t\epsilon[t_o + t_d, t_f] \\ \dot{\lambda}_2(t) &= -\lambda_1(t) + 2\zeta\lambda_2(t), \qquad t\epsilon[t_o + t_d, t_f] \end{aligned} \tag{5.28}$$

$$\lambda_1(t_f) = \nu_1, \qquad \lambda_2(t_f) = \nu_2 \tag{5.29}$$

$$v_{op}(t,\nu,\lambda) = -\mathrm{sgn}(\lambda_2(t)), \qquad t\epsilon[t_o + t_d, t_f] \tag{5.30}$$

$$\nu_2 v(t_f) = -1 \tag{5.31}$$

Thus, $v_{op}(t)$ is determined once $\lambda_2(t)$ is known as a function of e_o, $\dot{e}_o$, and t. To accomplish this, however, t_f must be determined from Equation (5.31) and $\nu^T = [\nu_1, \nu_2]$ must be found such that Equation (5.24) is satisfied. Even for this simple problem, this is not an easy task.

The minimum-time switching curves for the analogous delay-free optimization problem were described in Section 2.2.2. To simplify the discussion here, assume $\zeta = 0.0$. Also assume that t_d is of such a magnitude that no double commands occur, otherwise it would not be possible to obtain a closed-loop, feedback control law for this optimization problem (see Section 2.2.1). From the discussion in Section 5.1, this delay-free solution can be used to construct the command curves which give optimal trajectories for the time-delay system.

This construction is accomplished by first observing that $\Delta\alpha = -\Delta t$ in the $(e,\dot{e})$ - phase plane (see Figure 2.1). A clockwise rotation on a trajectory about $(\pm N,0)$ of t_d radians therefore represents prediction of the state a time t_d into the future. It was concluded in Section 5.1 that after time $t_o + t_d$ the optimal trajectories of the delay-free and time-delay systems are identical. Also recall that switching occurs a time t_d after the command has been given. The optimal command curve is thus obtained by rotating the $(u = -N$ to $u = +N)$ - switching curve t_d radians counterclockwise about $(-N,0)$ and by rotating the $(u = +N$ to $u = -N)$ - switching curve t_d radians counterclockwise about $(+N,0)$. This location of the command curve insures that the switching curves for the optimal delay-free and time-delay systems are the same, provided that no disturbance acts between command and switch. The optimal command curves, along with a sample trajectory, are illustrated in Figure 5.3.

Notice that since the optimal delay-free switching curve was split at the origin to form the command curve, the resulting command curve is disconnected. The optimal command curve which connects the two segments is found by recalling that $v(t) = 0,\ t > t_f$, since the initial state is being transferred to the origin. Since the command $v = 0$ must be given at time $t_f - t_d$, the connecting command curve must be the locus of all points in state space having a minimum-settling-time of magnitude t_d. The calculation of this locus for general linear systems with scalar control is performed in Appendix B with this second-order system taken as a specific example. When $t_d = \frac{\pi}{2}$ radians, the locus is given by

$$(\dot{e}_L \mp N)^2 + (e_L \mp N)^2 = (2N)^2. \tag{5.31}$$

This zero-command curve is indicated by a dashed line in Figure 5.3. It is noted that this curve is an isochrone of the origin with an associated time of magnitude t_d.

It is concluded that any initial state, which places $x(t_o + t_d)$ outside of the region bounded by the zero-command curve, is transferred to the origin in minimum time by the command curve described above and

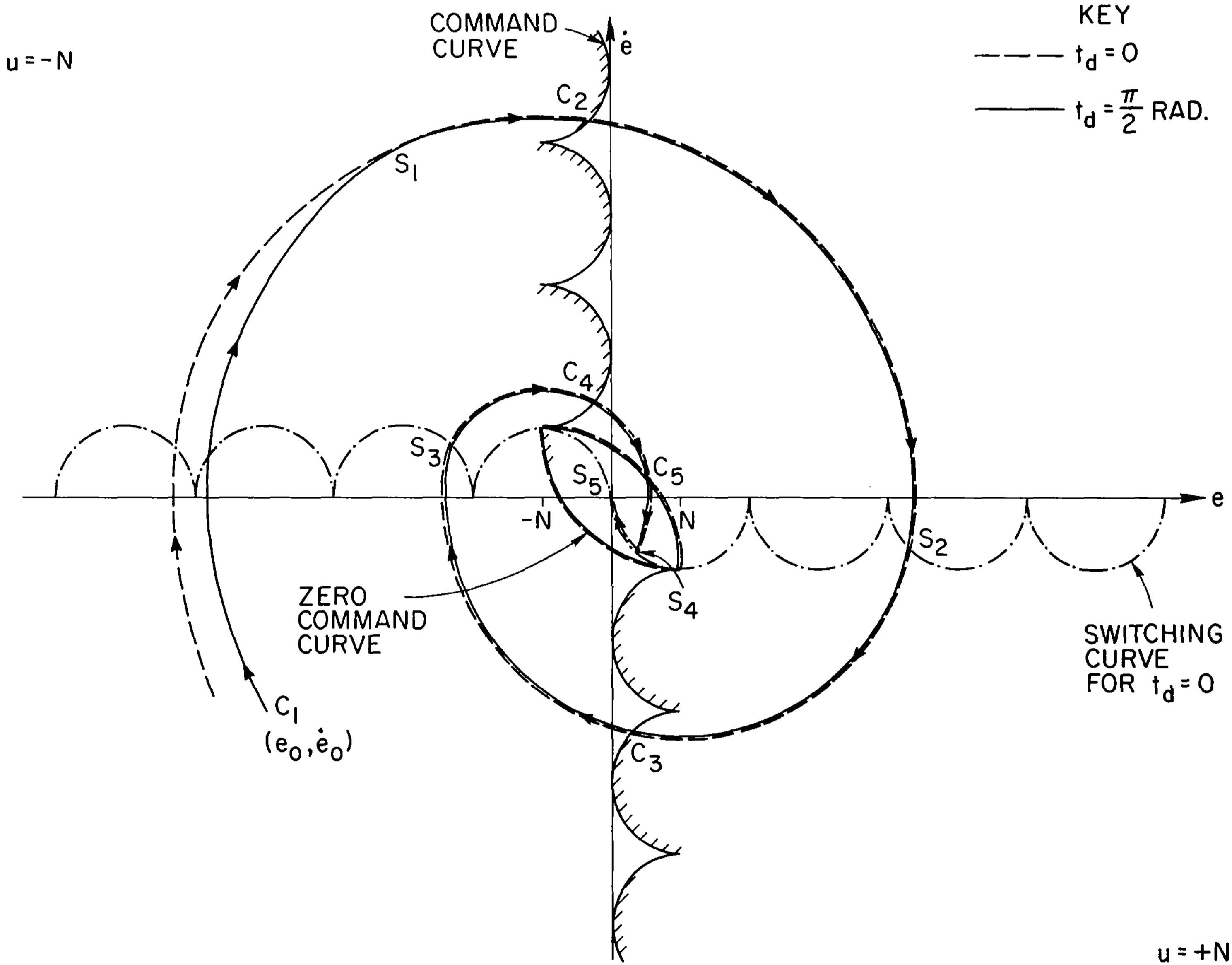


Figure 5.3. Minimum-Time Control to the Origin of the 2^{nd}-Order System for Zero Damping and a Time Delay which Gives No Double Commands.

shown in Figure 5.3 for $t_d = \frac{\pi}{2}$ radians. If multiple commands occur, a feedback solution does not appear to be possible since the command lines become a function of the initial state. Similarly, if $x(t_o + t_d)$ lies within the region bounded by the zero-command curve, the command lines are functions of the initial state and thus a feedback control law does not appear to be possible.

Several control alternatives are available to handle the case when $x(t_o + t_d)$ lies within the region bounded by the zero-command curve:

1) The minimum-time trajectory can be generated in an open loop control scheme by predicting ahead until t_f and issuing commands such that switching occurs on the optimal switching curve of the delay-free system.

2) If N is sufficiently small compared to acceptable values of e and $\dot{e}$, the region bounded by the zero command curve can define a "dead zone region" within which $v = 0$. That is, we could allow state deviations that remain inside the region, but transfer deviations outside the region at time $t_o + t_d$ to the origin in minimum time. In some sense, this would increase the terminal manifold from a point to a region in state space.

3) A dual control system could be used with the zero-command curve acting as the switching curve between the two control schemes. For state deviations outside this switching curve, the minimum-time control scheme discussed above is used. Inside the switching curve a controller such as that derived in Section 5.2.2 can be used. The final time would not have to be greater than $2t_d$ since the entire region is bounded by points whose minimum-settling-time is t_d. Since the region is small, proper values of Q_1, Q_2, and t_f could be chosen so that $x(t_f)$ is sufficiently small and the control bounds are not exceeded.

The three control schemes suggested above could, theoretically, be applied to any control problem for which the theory developed here is applicable. The zero-command curve can always be generated from Equation (B5) in Appendix B. The geometrical description of these command curves becomes much more difficult, however, for more complicated problems. The theory and ideas which were applied to solve this second-order example are still applicable to higher-order systems, but the lack of knowledge of the switching curves for the analogous delay-free systems limits our ability to solve the higher-order time-delay optimization problems.

CHAPTER VI

SUMMARY AND CONCLUSIONS

In this study, two analogous systems and optimization problems were defined and analyiz製. The only distinction between the two systems is that one possesses a large time delay in the control. It was shown that systems with time delay in the control are uncontrollable from the present time, t_o, until time $t_o + t_d$. This fact allowed us to formulate the optimization problem for the time-delay system in such a way that the necessary conditions for optimality of the delay-free system become applicable to the time-delay system. The optimization theory presented here culminates in the statement of Theorem 4.2, which is considered to be the main result of this study.

The necessary conditions for optimality of time-delay systems can be obtained by means of the calculus of variations, as demonstrated in Appendix A, without ever considering the analogous delay-free system. These conditions, in certain instances, can then be used directly to derive an optimal control law for the time-delay system. Several n^{th}-order example problems, with carefully chosen performance criterions, were presented where this was the procedure which was followed.

By establishing and considering the delay-free analogue of the time-delay system, we have shown that knowledge of the solution of one of the optimization problems is sufficient to solve the analogue optimization problem. A second-order problem with bounded control is used to demonstrate this principle. The creation of the zero-command curve was the main feature in this problem. It was demonstrated by example that large amplitude limit cycles are characteristic of time-delay systems with bang-bang control. The zero-command curve not only eliminates steady state limit cycles, it also insures that the system trajectories end at the origin when the state at time $t_o + t_d$ lies outside of the region bounded by the zero-command curve. Several control alternatives are suggested which could be used to handle the case when $x(t_o + t_d)$ lies inside this region.

Obtaining the feedback control law for this second-order example was accomplished only by assuming the non-existence of double commands. This assumption places an upper limit upon the magnitude which the time delay can assume, while still requiring that the optimal control law have a feedback structure. When double commands do occur, the optimal control law can still be generated, but only in an open-loop sense.

Several generalizations of this work are possible. First, it was assumed throughout this study that the system dynamics are linear. This assumption was convenient in the controllability discussion, but investigation of the necessary conditions for optimality reveals that system linearity is not essential. The optimization problem was posed by first considering the results of the controllability discussion. Once posed, however, the optimization problem became independent of system linearity. Implementation, however, of the resulting control laws, becomes exceedingly difficult, as is the case with non-linear, delay-free optimal control problems. To generalize this study to encompass non-linear systems, one would have to consider controllability of non-linear systems to see if a given optimization problem statement is meaningful.

Only a regulator type control problem was considered in this work. This was done merely to simplify the analysis. Obtaining the necessary conditions for optimality did not depend upon the type of control problem considered. Only in the use of these necessary conditions to obtain an optimal control law does the regulator assumption simplify the analysis.

Finally, the assumptions that the time delay was constant with respect to time and identical in magnitude for each component of the control vector arose from the model problem which motivated this research.* To remove these assumptions would require further analysis, but it is felt that the ideas discussed in this paper can be utilized when considering this more complicated problem.

To conclude, it can be said that the ability to control optimally the class of systems considered in this paper is dependent upon the ability to control optimally the delay-free analogue and the ability to predict the disturbances which act on the system. The implementation problems will be at least as difficult as the implementation problems

*Attitude control, from earth, of a deep-space satellite approximately satisfies these assumptions.

encountered in delay-free controllers. The need to predict the future state is the result of the inherent period of uncontrollability possessed by linear systems with a time delay in the control.

APPENDIX A

DERIVATION OF NECESSARY CONDITIONS FOR OPTIMALITY OF SYSTEMS WITH TIME DELAY

The purpose of this appendix is to derive first-order necessary conditions for minimizing the augmented performance index, $\bar{J}$, for systems with time delay in the control (Equation (4.24)):

$$\bar{J} = [F[x(t),t] + \nu^T\psi[x(t),t]]_{t = t_f} + \int_{t_o + t_d}^{t_f} \left\{ H(x,\lambda,v,t) - \lambda^T(t)\dot{x}(t) \right\} dt \tag{A1}$$

To accomplish the minimization of $\bar{J}$, we take the differential of Equation (A1), remembering that t_f may be free:

$$d\bar{J} = [(\frac{\partial\Phi}{\partial t} + L)dt + \frac{\partial\Phi}{\partial x}dx]_{t = t_f} + \int_{t_o + t_d}^{t_f} (\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial v}\delta v - \lambda^T(t)\delta\dot{x})dt \tag{A2}$$

where

$$\Phi = F[x(t),t] + \nu^T\psi[x(t),t]. \tag{A3}$$

Now integrate Equation (A2) by parts and use $\dot{\delta x} = dx - \dot{x}dt$ (see [12], pg. 72) to obtain

$$d\bar{J} = (\frac{\partial\Phi}{\partial t} + L + \lambda^T\dot{x})_{t=t_f}dt_f + (\lambda^T\delta x)_{t=t_o + t_d} + [(\frac{\partial\Phi}{\partial x} - \lambda^T)dx]_{t = t_f} + \int_{t_o + t_d}^{t_f} [(\frac{\partial H}{\partial x} + \dot{\lambda}^T)\delta x + \frac{\partial H}{\partial v}\delta v]dt \tag{A4}$$

Now for $\bar{J}$ to be minimized, it is necessary that $d\bar{J}$ be zero (or as close to zero as possible). Therefore, since dt_f, $\delta x(t)$, and $dx(t_f)$ are arbitrary variations, choose the coefficients of these terms to be zero in Equation (A4):

$$\left[\frac{\partial\Phi}{\partial t} + L + \lambda^T(Ax + Bv)\right]_{t = t_f} = 0 \tag{A5}$$

$$\lambda^T(t_f) = \frac{\partial\Phi}{\partial x}\bigg|_{t = t_f} \tag{A6}$$

$$\dot{\lambda}^T(t) = -\frac{\partial H}{\partial x}, \quad t\epsilon[t_o + t_d, t_f] \tag{A7}$$

Equations (A5)-(A7) are equivalent to Equation (4.30), Equation (4.29), and Equation (4.26), respectively. Since $x(t_o + t_d)$ is assumed specified, $\delta x(t_o + t_d)$ is zero. Hence, $d\bar{J}$ reduces to

$$d\bar{J} = \int_{t_o + t_d}^{t_f} \frac{\partial H}{\partial v}\delta v \, dt \tag{A8}$$

Now, to minimize $\bar{J}$, Pontryagin et.al [13] showed that $v(t)$ must satisfy the following relationship over the time interval of the integral in Equation (A8):

$$v_{op}(t) = \underset{v\epsilon U}{\arg\min}\, H(x,\lambda,v,t), \quad t\epsilon[t_o + t_d, t_f] \tag{A9}$$

Equation (A9) is precisely Equation (4.27). If v were unbounded, setting $H_v = 0$ in Equation (A8) for $t\epsilon[t_o + t_d, t_f]$ insures that we have found an extremum of $\bar{J}$. Second-order sufficient conditions would be required in order to determine if the extremum is a minimum. When v is bounded, the solution of Equation (A9) gives $v_{op}(t)$. If $v_{op}(t)$ lies on a boundary of admissible v, we are assured that $\bar{J}$ has been minimized. When $v_{op}(t)$ does not lie on a boundary of admissible v, the second-order sufficient conditions are again required to determine whether $\bar{J}$ has been minimized.

APPENDIX B

CALCULATION OF ZERO-COMMAND LOCUS FOR LINEAR SYSTEMS AND MINIMUM-TIME TRAJECTORIES

We have assumed the non-existence of double-commands in Section 5.3. It is reasonable to assume, therefore, that at most one switch occurs from time $t_f - t_d$ to time t_f, assuming scalar control. Designate the points lying on the locus of zero-command by x_L. The control is $u = \pm N$ and the system dynamics are given by

$$\dot{x}(t) = A(t)x(t) + B(t)(\pm N). \tag{B1}$$

The solution of Equation (B1) can be written

$$x(t) = \varphi(t,t_o)x_L + \int_{t_o}^{t} \varphi(t,\tau)B(\tau)(\pm N)d\tau \tag{B2}$$

Let time $t_o + t_1$ be the time of the switch point. Then the switch point is given by

$$x(t_o + t_1) = \varphi(t_o + t_1,t_o)x_L + \int_{t_o}^{t_o + t_1} \varphi(t_1 + t_o,\tau)B(\tau)(\pm N)d\tau \tag{B3}$$

and the final state is written

$$\begin{aligned} x(t_f) = 0 = {} & \varphi(t_o + t_d,t_o + t_1)x(t_o + t_1) \\ & + \int_{t_o+t_1}^{t_o+t_d} \varphi(t_o + t_d,\tau)B(\tau)(\mp N)d\tau \end{aligned} \tag{B4}$$

Now substitute $x(t_o + t_1)$ from Equation (B3) into Equation (B4), assuming t_o is zero for simplicity. Solving for x_L then gives

$$\frac{x_L}{N} = \pm \left[\int_{t_1}^{t_d} \varphi(o,\tau)B(\tau)d\tau - \int_{o}^{t_1} \varphi(o,\tau)B(\tau)d\tau \right], \quad o \leq t_1 \leq t_d \tag{B5}$$

Thus, considering x_L as a function of t_1, Equation (B5) with t_1 varying from zero to t_d generates the locus of points with minimum-time trajectories which pass thru the origin at time t_d.

As a specific example, consider the second-order system of Section 5.3. For this system,

$$\varphi(o,\tau) = \begin{bmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{bmatrix}, \quad B(\tau) = \begin{bmatrix} o \\ 1 \end{bmatrix}. \tag{B6}$$

Substituting Equation (B6) into Equation (B5) gives

$$\frac{x_L}{N} = \pm \begin{bmatrix} \cos t_d + 1 - 2\cos t_1 \\ \sin t_d - 2\sin t_1 \end{bmatrix} = \frac{1}{N} \begin{bmatrix} e_L \\ \dot{e}_L \end{bmatrix} \tag{B7}$$

Eliminating t_1 in Equation (B7) finally gives

$$(\dot{e}_L \mp N \sin t_d)^2 + (e_L \mp N \mp N\cos t_d)^2 = (2N)^2 \tag{B8}$$

as the desired locus of points. The points $(e_L, \dot{e}_L)$ in the phase plane thus lie on circular arcs of radius $2N$ and center $(\pm N \pm N\cos t_d, \pm N\sin t_d)$. Note that if $\zeta \neq 0$ in the second-order example, the circular arcs would become arcs of logarithmic spirals, but the structure of the locus would be the same.

REFERENCES

1. Albert E. Sabroff, "Advanced Spacecraft Stabilization and Control Techniques", Journal of Spacecraft and Rockets, December, 1968, Volume 5, Number 12, pages 1377-1393.

2. James Adams, "An Investigation of the Effects of Time Lag Due to Long Transmission Distances Upon Remote Control, Phase I Tracking Experiments", NASA TN D-1211, 1961.

3. Charles F. Martin, "Investigation of a Remotely Controlled Lunar Surface Vehicle Utilizing Bang-Bang Control", Engineer's Thesis, Department of Applied Mechanics, Stanford University, July 1968.

4. I. Flügge-Lotz, Discontinuous Automatic Control, Princeton University Press, Princeton, New Jersey, 1953.

5. A.T. Fuller, "Optimal Nonlinear Control of Systems with Pure Delay", Int. J. Control, 1968, Volume 8, Number 2, pages 145-168.

6. K. Ichikawa, "Pontryagin's Maximum Principle in Optimizing Time-Delay Systems", Electrical Engineering in Japan, Volume 87, December 1967, pages 75-83.

7. I. Flügge-Lotz, Discontinuous and Optimal Control, McGraw-Hill Book Company, New York, 1968.

8. D. Bushaw, "Optimal Discontinuous Forcing Terms", Contributions to the Theory of Nonlinear Oscillations, Volume IV, Princeton University Press, Princeton, New Jersey, 1958.

9. R.E. Kalman, P.L. Falb, and M.A. Arbib, Topics in Mathematical System Theory, McGraw-Hill Book Company, New York, 1969.

10. R. E. Kalman, Y. C. Ho, and K. S. Narendra, "Controllability of Linear Dynamical Systems", Contributions to Differential Equations, Volume 1, Number 2, Wiley, 1961.

11. E. B. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley and Sons, Inc., New York, 1967.

12. A. E. Bryson, Jr., and Yu-Chi Ho, Applied Optimal Control: Optimization, Estimation, and Control, Blaisdell Publishing Company, 1968.

13. L. S. Pontryagin, V. G. Boltianskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience Publishers, Inc., New York, 1962.

NASA-Langley, 1970 —— 19 CR-1553